LECTURE NOTES

ESB2024 Precourse: Constitutive modelling of soft tissues

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1 Preliminaries

1.1 The indicial notation and summation convention

In a real three-dimensional vector space $\mathscr{V}$ with *basis* vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ a vector $\mathbf{v}$ may be written

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3, \tag{1.1}$$

where (v_1, v_2, v_3) are the *components* of $\mathbf{v}$ with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Here, we take $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to be *orthonormal* (i.e. mutually orthogonal unit vectors), and this condition may be written compactly in the form

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j, \end{cases} \tag{1.2}$$

where δ_{ij} is the Kronecker delta, defined by the right-hand equality in (1.2). Note that the notation $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is often used for the basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ but this notation is unsuitable for our purposes.

When it is understood that the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is being used then the vector $\mathbf{v}$ may be represented by the triad of components (v_1, v_2, v_3) or, more compactly, simply as v_i, where it the *index* (or *suffix*) i takes values in the set $\{1, 2, 3\}$.

Letters $i, j, k, \ldots$ are used as indices; each takes values in the set $\{1, 2, 3\}$.

We now write (1.1) as

$$\mathbf{v} = \sum_{i=1}^{3} v_i\mathbf{e}_i. \tag{1.3}$$

In shorthand notation (the *summation convention*) we rewrite (1.3) as

$$\mathbf{v} = v_i\mathbf{e}_i. \tag{1.4}$$

By convention, summation over any index (in this case i) from 1 *to* 3 *is implied by repetition of that index.*

Any index over which there is summation is known as a *dummy index* (because any choice of letter to represent the summation will do). Thus, $v_i\mathbf{e}_i = v_j\mathbf{e}_j = v_p\mathbf{e}_p$, for example.

Examples

1. $a_ia_i = a_1^2 + a_2^2 + a_3^2$

2. $a_pb_p = a_1b_1 + a_2b_2 + a_3b_3$

Any index over which there is no summation is called a *free* index. Thus, on its own, a_i represents the vector $\mathbf{a}$, and

$$a_p b_p a_i,$$

where p is a dummy index and i is a free index, represents the vector $(a_p b_p)\mathbf{a}$ with $a_p b_p$ being just a number [see exercise (ii) below].

Examples

(i) $\delta_{ij} b_j = b_i$

(ii) $\mathbf{a} \cdot \mathbf{b} = (a_i \mathbf{e}_i) \cdot (b_j \mathbf{e}_j) = a_i b_j \mathbf{e}_i \cdot \mathbf{e}_j = a_i b_j \delta_{ij} = a_i b_i$

1.2 The alternating symbol

Now consider $\mathbf{a} \times \mathbf{b} = (a_i \mathbf{e}_i) \times (b_j \mathbf{e}_j) = a_i b_j \mathbf{e}_i \times \mathbf{e}_j$ and note that

$$\mathbf{e}_i \times \mathbf{e}_j = \begin{cases} \mathbf{0} & \text{if } i = j \\ +\mathbf{e}_k & \text{if } (ijk) \text{ is an even permutatino of } (123) \\ -\mathbf{e}_k & \text{if } (ijk) \text{ is an odd permutation of } (123). \end{cases}$$

This prompts us to write $\mathbf{e}_i \times \mathbf{e}_j$ as a linear combination of the basis vectors in the form

$$\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk} \mathbf{e}_k, \tag{1.5}$$

noting that there is summation over the repeated index k, where the *alternating symbol* ϵ_{ijk} is defined by

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (ijk) \text{ is an even perm of } (123) \\ -1 & \text{if } (ijk) \text{ is an odd perm of } (123) \\ 0 & \text{otherwise} \end{cases} \tag{1.6}$$

The properties

$$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki},$$

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}$$

follow immediately from the definition (1.6). Note, in particular, that ϵ_{ijk} is antisymmetric in any pair of indices.

Returning to the calculation of $\mathbf{a} \times \mathbf{b}$ we now obtain

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= a_i b_j (\epsilon_{ijk} \mathbf{e}_k) \\ &= \epsilon_{kij} a_i b_j \mathbf{e}_k \\ &= \epsilon_{1ij} a_i b_j \mathbf{e}_1 + \epsilon_{2ij} a_i b_j \mathbf{e}_2 + \epsilon_{3ij} a_i b_j \mathbf{e}_3 \\ &= (a_2 b_3 - a_3 b_2)\mathbf{e}_1 + (a_3 b_1 - a_1 b_3)\mathbf{e}_2 + (a_1 b_2 - a_2 b_1)\mathbf{e}_3 \end{aligned}$$

carrying out first the summation over k then the summations over i and j together using the properties of (1.6).

In index notation we may write the k component of $\mathbf{a} \times \mathbf{b}$ as $\epsilon_{kij} a_i b_j$, for example.

Examples

1. The triple scalar product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_k (\mathbf{b} \times \mathbf{c})_k = a_k (\epsilon_{kij} b_i c_j) = \epsilon_{kij} a_k b_i c_j.$$

By carrying out the summations over i, j, k it can be seen that this is equivalent to the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

from which the symmetries

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$$

may be deduced.

2. The identity

$$\epsilon_{kij} \epsilon_{kpq} = \delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}. \tag{1.7}$$

Since $\epsilon_{kij} \epsilon_{kpq} = \epsilon_{1ij} \epsilon_{1pq} + \epsilon_{2ij} \epsilon_{2pq} + \epsilon_{3ij} \epsilon_{3pq}$ we see from the definition (1.6) that

$$\epsilon_{kij} \epsilon_{kpq} = \begin{cases} 0 \ \text{ if } i = j \text{ or } p = q \\ 0 \ \text{ if } i \neq j, p \neq q \text{ and } pq \neq ij \text{ or } ji \\ +1 \ \text{ if } pq = ij \ (i \neq j) \\ -1 \ \text{if } pq = ji \ (i \neq j) \end{cases}$$

This covers all possible combinations of $ijpq$. The expression $\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$ takes precisely the same values in each of the various cases. Hence (1.7) is established. Check by considering specific values of $ijpq$. A better proof of the identity (1.7) will be provided later in the context of isotropic tensor theory.

3. The triple vector product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

$$\begin{aligned} [\mathbf{a} \times (\mathbf{b} \times \mathbf{c})]_i &= \epsilon_{ijk} a_j (\mathbf{b} \times \mathbf{c})_k \\ &= \epsilon_{kij} a_j (\epsilon_{kpq} b_p c_q) \\ &= \epsilon_{kij} \epsilon_{kpq} a_j b_p c_q \\ &= (\delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}) a_j b_p c_q \\ &= a_j \delta_{ip} b_p \delta_{jq} c_q - a_j \delta_{jp} b_p \delta_{iq} c_q \\ &= a_j b_i c_j - a_j b_j c_i \\ &= (a_j c_j) b_i - (a_j b_j) c_i \\ &= [(\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}]_i. \end{aligned}$$

Since this holds for each index i we deduce the standard identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

4. The curl of a vector function in index notation

This is defined by

$$[\nabla \times \mathbf{F}]_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} F_k \tag{1.8}$$

analogously to the cross product of two vectors.

Note that the identity (1.7) can be used whenever two cross products occur, as shown in the example below.

5. Prove the identity $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$

$$\begin{aligned}
[\nabla \times (\nabla \times \mathbf{F})]_i &= \epsilon_{ijk} \frac{\partial}{\partial x_j} (\nabla \times \mathbf{F})_k \\
&= \epsilon_{ijk} \frac{\partial}{\partial x_j} \left(\epsilon_{kpq} \frac{\partial}{\partial x_p} F_q \right) \\
&= \epsilon_{kij} \epsilon_{kpq} \frac{\partial^2 F_q}{\partial x_j \partial x_p} \\
&= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) \frac{\partial^2 F_q}{\partial x_j \partial x_p} \\
&= \frac{\partial^2 F_j}{\partial x_j \partial x_i} - \frac{\partial^2 F_i}{\partial x_j \partial x_j} \\
&= \frac{\partial}{\partial x_i} \left(\frac{\partial F_j}{\partial x_j} \right) - \frac{\partial^2 F_i}{\partial x_j \partial x_j} \\
&= \frac{\partial}{\partial x_i} (\nabla \cdot \mathbf{F}) - \nabla^2 F_i \\
&= [\nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}]_i
\end{aligned}$$

and hence the required identity.

2 Cartesian tensors

2.1 Change of basis

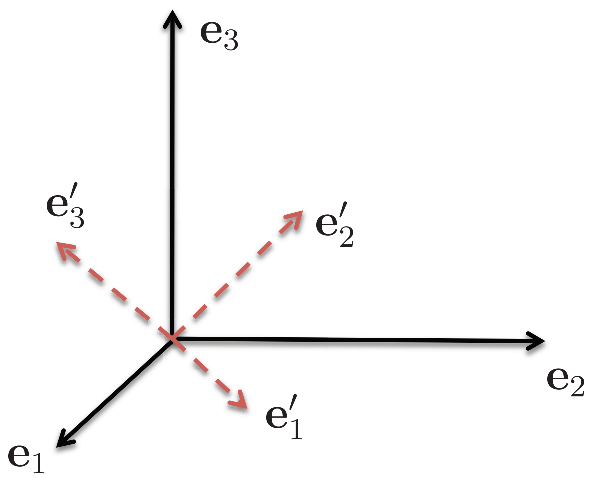
Let $\mathscr{V}$ denote a three-dimensional (Euclidean) vector space, and select right-handed orthonormal basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ for $\mathscr{V}$ so that

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j. \end{cases} \tag{2.1}$$

Then, in the summation convention, any vector $\mathbf{v}$ can be written as

$$\mathbf{v} = v_i \mathbf{e}_i \tag{2.2}$$

with respect to the chosen basis.



Now choose a second orthonormal basis $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$. Again, we have the connections

$$\mathbf{e}'_i \cdot \mathbf{e}'_j = \delta_{ij}, \tag{2.3}$$

and we may also decompose $\mathbf{v}$ with respect to this new basis as

$$\mathbf{v} = v'_i \mathbf{e}'_i. \tag{2.4}$$

Since $\mathscr{V}$ has three dimensions, $\mathbf{e}'_i$ must be expressible as a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ for each $i \in \{1, 2, 3\}$. We write this in the form

$$\mathbf{e}'_i = l_{ij} \mathbf{e}_j \qquad (i \in \{1, 2, 3\};\ \text{sum over } j). \tag{2.5}$$

It follows from (2.5) that

$$l_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j, \tag{2.6}$$

which are the so-called *direction cosines* of $\mathbf{e}'_i$ relative to $\mathbf{e}_j$.

From (2.3) and (2.5) we have

$$\delta_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j = (l_{ik}\mathbf{e}_k) \cdot \mathbf{e}'_j = l_{ik}(\mathbf{e}'_j \cdot \mathbf{e}_k) = l_{ik}l_{jk},$$

and similarly

$$l_{ki}l_{kj} = \delta_{ij}.$$

Thus

$$l_{ik}l_{jk} = \delta_{ij} = l_{ki}l_{kj}. \tag{2.7}$$

In matrix notation, defined by $L \equiv (l_{ij})_{i,j=1,2,3} \in M_{3\times3}(\mathbb{R}),\ I \equiv (\delta_{ij})_{i,j=1,2,3} \in M_{3\times3}(\mathbb{R})$, equations (2.7) may be expressed as

$$LL^T = I = L^T L, \tag{2.8}$$

from which it follows that $\det(LL^T) = 1$ and hence $\det L = \pm 1$. Thus, L is an *orthogonal* matrix. If $\det L = +1$ then L is said to be *proper orthogonal*, and it represents a rotation, while if $\det L = -1$ then L is said to be *improper orthogonal*, and it represents a rotation combined with a reflection.

We now have

$$l_{ik}\mathbf{e}'_i = l_{ik}l_{ij}\mathbf{e}_j = \delta_{jk}\mathbf{e}_j = \mathbf{e}_k,$$

and, combining this with (2.5), we obtain the connections

$$\mathbf{e}'_i = l_{ij}\mathbf{e}_j, \quad \mathbf{e}_i = l_{ji}\mathbf{e}'_j. \tag{2.9}$$

Also, since

$$\mathbf{v} = v_j\mathbf{e}_j = v_j l_{ij}\mathbf{e}'_i, \quad \mathbf{v} = v'_i\mathbf{e}'_i$$

we obtain $v'_i = l_{ij}v_j$, and, similarly, $v_i = l_{ji}v'_j$. Thus, *the components of a vector transform in the same way as the basis vectors under a change of basis* — compare (2.9) with

$$v'_i = l_{ij}v_j, \quad v_i = l_{ji}v'_j. \tag{2.10}$$

Examples

1. If the origin is fixed then the components of the position vector $\mathbf{x}$ transform according to

$$x'_i = l_{ij}x_j, \quad x_j = l_{ij}x'_i,$$

and it follows that

$$l_{ij} = \frac{\partial x'_i}{\partial x_j} = \frac{\partial x_j}{\partial x'_i}.$$

2. The transformation of basis vectors defined by

$$L = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

leaves $\mathbf{e}_3$ unchanged and represents an anticlockwise rotation through an angle θ in the $(1, 2)$-plane.

3. The transformation of basis vectors defined by

$$L = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha & 0 \\ \sin 2\alpha & -\cos 2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

represents a rotation through an angle 2α about $\mathbf{e}_3$ followed by a reflection in $\mathbf{e}'_1$ in the $(1, 2)$-plane.

2.2 Definition of a Cartesian tensor

The following definition relates to the three-dimensional space $\mathscr{V}$, but is equally applicable to vector spaces of higher dimension.

A *second-order tensor* $\mathbf{T}$ is a linear mapping $\mathbf{T} : \mathscr{V} \to \mathscr{V}$.

If we consider V in conjunction with the set of all possible right-handed orthonormal basis vectors related by (2.9) with L *proper orthogonal* then $\mathbf{T}$ is referred to as a *Cartesian tensor*, and its components T_{ij} and T'_{ij} relative to the bases $\{\mathbf{e}_i\}$ and $\{\mathbf{e}'_i\}$ respectively are related by

$$T'_{ij} = l_{ip} l_{jq} T_{pq}. \tag{2.11}$$

The transformation rule (2.11) can be established as follows. Let $\mathbf{u} \in \mathscr{V}$ and let $\mathbf{v} = \mathbf{T}\mathbf{u} \in \mathscr{V}$, so that

$$v_i = T_{ij} u_j, \quad v'_i = T'_{ij} u'_j$$

with respect to bases $\{\mathbf{e}_i\}$ and $\{\mathbf{e}'_i\}$ respectively. But, since $\mathbf{u}$ and $\mathbf{v}$ are vectors, we have

$$v'_i = l_{ip} v_p, \quad u'_j = l_{jq} u_q.$$

Hence

$$l_{ip} T_{pq} u_q = l_{ip} v_p = T'_{ij} l_{jq} u_q,$$

which may be re-written as

$$(l_{ip} T_{pq} - T'_{ij} l_{jq}) u_q = 0.$$

This must hold for all choices of u_q. Hence

$$l_{ip} T_{pq} - T'_{ip} l_{pq} = 0.$$

Multiplication of this by l_{jq} then gives

$$l_{ip}l_{jq}T_{pq} - T'_{ip}(l_{pq}l_{jq}) = 0,$$

i.e.

$$T'_{ij} = l_{ip}l_{jq}T_{pq},$$

as required.

In matrix notation, we write

$$[\mathbf{T}]' = L[\mathbf{T}]L^T.$$

Analogously to (2.11), a *Cartesian tensor of order* n – in brief CT(n) – has components $T_{ijk\ldots}$ (with n indices $i, j, k, \ldots$) which transform according to the rule

$$T'_{ijk\ldots} = l_{ip}l_{jq}l_{kr}\ldots T_{pqr\ldots} \tag{2.12}$$

under a change of orthonormal basis. [More generally, $\mathbf{T}$ is referred to as *multilinear mapping* over $\mathscr{V}$, but we shall not use this terminology in this course.]

Examples

1. A scalar is CT(0): $\alpha' = \alpha$.

2. A vector is CT(1): $v'_i = l_{ij}v_j$.

3. The *dyadic* (or *tensor*) *product* of two vectors $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} \otimes \mathbf{v}$, is CT(2). Its components are defined by $(\mathbf{u} \otimes \mathbf{v})_{ij} = u_i v_j$, and the transformation rule is simply

$$(\mathbf{u} \otimes \mathbf{v})'_{ij} = u'_i v'_j = l_{ip}u_p l_{jq}v_q = l_{ip}l_{jq}u_p v_q = l_{ip}l_{jq}(\mathbf{u} \otimes \mathbf{v})_{pq}.$$

4. The Kronecker delta is CT(2):

$$\begin{aligned} \delta'_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j &= (l_{ip}\mathbf{e}_p) \cdot (l_{jq}\mathbf{e}_q) \\ &= l_{ip}l_{jq}(\mathbf{e}_p \cdot \mathbf{e}_q) \\ &= l_{ip}l_{jq}\delta_{pq} \ (= \delta_{ij}) \end{aligned}$$

Note that the components of the Kronecker delta are the same for *all* (orthonormal) bases. The Kronecker delta is an example of an *isotropic tensor*. Isotropic tensors will be discussed later.

2.3 The tensor product notation

The tensor product notation has been defined in Example 3 above. An alternative definition (without reference to choice of basis) is as follows. For a given pair of vectors $\mathbf{u}$ and $\mathbf{v}$ the tensor product $\mathbf{u} \otimes \mathbf{v}$ is defined by the identity

$$(\mathbf{u} \otimes \mathbf{v})\mathbf{a} = (\mathbf{v} \cdot \mathbf{a})\mathbf{u} \qquad \text{for all } \mathbf{a} \in \mathscr{V}. \tag{2.13}$$

The component form of (2.13) with respect to the basis $\{\mathbf{e}_i\}$ is

$$(\mathbf{u} \otimes \mathbf{v})_{ij} a_j = v_j a_j u_i \qquad \text{for all } \mathbf{a} \in \mathscr{V},$$

and this is equivalent to the definition given in Example 3.

Important note: in general, a CT(2) *cannot* be represented as the tensor product of two vectors, i.e. a given $\mathbf{T}$ cannot be written in the form $\mathbf{u} \otimes \mathbf{v}$ for any pair of vectors $\mathbf{u}, \mathbf{v} \in \mathscr{V}$. However, $\mathbf{T}$ can always be written as a linear combination of products of the form $\mathbf{u} \otimes \mathbf{v}$; the reason for this is as follows.

Note that if α and β are scalars and $\mathbf{S}$ and $\mathbf{T}$ are CT(2) then $\alpha\mathbf{S} + \beta\mathbf{T}$ is also CT(2). It follows that the set of all CT(2)s forms a vector space (*not* the same as $\mathscr{V}$). This space has dimension $9 = 3^2$ (note that T_{ij} has 9 components — $T_{11}, T_{12}, T_{13}, \ldots, T_{33}$). A basis 'vector' for this 9-dimensional space can be written as $\mathbf{e}_i \otimes \mathbf{e}_j$ — there are 9 such objects altogether:

$$\begin{array}{ccc}
\mathbf{e}_1 \otimes \mathbf{e}_1 & \mathbf{e}_1 \otimes \mathbf{e}_2 & \mathbf{e}_1 \otimes \mathbf{e}_3 \\
\mathbf{e}_2 \otimes \mathbf{e}_1 & \mathbf{e}_2 \otimes \mathbf{e}_2 & \mathbf{e}_2 \otimes \mathbf{e}_3 \\
\mathbf{e}_3 \otimes \mathbf{e}_1 & \mathbf{e}_3 \otimes \mathbf{e}_2 & \mathbf{e}_3 \otimes \mathbf{e}_3.
\end{array}$$

Note, in particular, that $\mathbf{e}_1 \otimes \mathbf{e}_2 \neq \mathbf{e}_2 \otimes \mathbf{e}_1$.

Thus, $\mathbf{T}$ may be written in the form

$$\begin{aligned}
\mathbf{T} &= T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j \qquad \text{(sum over } i \text{ and } j\text{)} \\
&\equiv T_{11}\mathbf{e}_1 \otimes \mathbf{e}_1 + T_{12}\mathbf{e}_1 \otimes \mathbf{e}_2 + T_{13}\mathbf{e}_1 \otimes \mathbf{e}_3 \\
&\quad + T_{21}\mathbf{e}_2 \otimes \mathbf{e}_1 + T_{22}\mathbf{e}_2 \otimes \mathbf{e}_2 + T_{23}\mathbf{e}_2 \otimes \mathbf{e}_3 \\
&\quad + T_{31}\mathbf{e}_3 \otimes \mathbf{e}_1 + T_{32}\mathbf{e}_3 \otimes \mathbf{e}_2 + T_{33}\mathbf{e}_3 \otimes \mathbf{e}_3
\end{aligned} \tag{2.14}$$

If we choose a second basis $\mathbf{e}'_i \otimes \mathbf{e}'_j$ then we may also write

$$\mathbf{T} = T'_{ij}\mathbf{e}'_i \otimes \mathbf{e}'_j. \tag{2.15}$$

Consider the operation of $\mathbf{T}$ on a vector $\mathbf{u}$:

$$\begin{aligned}
\mathbf{T}\mathbf{u} &= (T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{u} && \text{by (2.14)} \\
&= T_{ij}(\mathbf{u} \cdot \mathbf{e}_j)\mathbf{e}_i && \text{by (2.13)} \\
&= T_{ij} u_j \mathbf{e}_i.
\end{aligned}$$

Thus,

$$(\mathbf{Tu})_i = T_{ij} u_j.$$

More particularly,

$$\begin{aligned}\mathbf{Te}_j &= (T_{pq}\mathbf{e}_p \otimes \mathbf{e}_q)\mathbf{e}_j \\ &= T_{pq}(\mathbf{e}_q \cdot \mathbf{e}_j)\mathbf{e}_p \\ &= T_{pq}\delta_{qj}\mathbf{e}_p = T_{pj}\mathbf{e}_p\end{aligned}$$

and hence

$$\mathbf{e}_i \cdot (\mathbf{Te}_j) = T_{pj}\mathbf{e}_i \cdot \mathbf{e}_p = T_{pj}\delta_{ip} = T_{ij}.$$

For a given $\mathbf{T}$ and choice of basis $\{\mathbf{e}_i\}$, therefore, we may *define* the components T_{ij} by

$$T_{ij} = \mathbf{e}_i \cdot (\mathbf{Te}_j) \equiv (\mathbf{Te}_j) \cdot \mathbf{e}_i. \tag{2.16}$$

2.4 The transpose of a CT(2)

The *transpose* $\mathbf{T}^T$ of $\mathbf{T}$ is defined via the identity

$$(\mathbf{Tu}) \cdot \mathbf{v} \equiv (\mathbf{T}^T\mathbf{v}) \cdot \mathbf{u} \qquad \text{for all } \mathbf{u}, \mathbf{v} \in \mathscr{V}. \tag{2.17}$$

Taking $\mathbf{u} = \mathbf{e}_j, \mathbf{v} = \mathbf{e}_i$ in (2.17) and using (2.16) we deduce that

$$T_{ij} = (\mathbf{T}^T)_{ji},$$

as might be expected. Note that

$$(\mathbf{a} \otimes \mathbf{b})^T = \mathbf{b} \otimes \mathbf{a}.$$

Definition

$$\mathbf{T} \text{ is } \textit{symmetric} \text{ if } \mathbf{T}^T = \mathbf{T}$$

$$\mathbf{T} \text{ is } \textit{skewsymmetric} \text{ or } \textit{antisymmetric} \text{ if } \mathbf{T}^T = -\mathbf{T}$$

Note that a symmetric CT(2) has 6 independent components and a skewsymmetric CT(2) has 3 independent components, and that any CT(2) $\mathbf{T}$ can be written as the sum of a symmetric and an antisymmetric part:

$$\mathbf{T} = \frac{1}{2}(\mathbf{T} + \mathbf{T}^T) + \frac{1}{2}(\mathbf{T} - \mathbf{T}^T). \tag{2.18}$$

2.5 The axial vector associated with a skewsymmetric CT(2)

If $\mathbf{W}$ is a skewsymmetric CT(2) then there exists a vector $\mathbf{w} \in \mathscr{V}$ (called the *axial vector* of $\mathbf{W}$) such that

$$\mathbf{Wv} = \mathbf{w} \times \mathbf{v} \qquad \text{for all } \mathbf{v} \in \mathscr{V}. \tag{2.19}$$

Proof. Since $\mathbf{W}$ is skewsymmetric ($W_{ij} = -W_{ji}$) it may be written

$$\mathbf{W} = W_{23}(\mathbf{e}_2 \otimes \mathbf{e}_3 - \mathbf{e}_3 \otimes \mathbf{e}_2) + W_{13}(\mathbf{e}_1 \otimes \mathbf{e}_3 - \mathbf{e}_3 \otimes \mathbf{e}_1) + W_{12}(\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1).$$

Then, using (2.13), we obtain

$$\begin{aligned}\mathbf{Wv} &= W_{23}v_3\mathbf{e}_2 - W_{23}v_2\mathbf{e}_3 + W_{13}v_3\mathbf{e}_1 - W_{13}v_1\mathbf{e}_3 + W_{12}v_2\mathbf{e}_1 - W_{12}v_1\mathbf{e}_2 \\ &= (W_{13}v_3 + W_{12}v_2)\mathbf{e}_1 + (-W_{12}v_1 + W_{23}v_3)\mathbf{e}_2 + (-W_{23}v_2 - W_{13}v_1)\mathbf{e}_3.\end{aligned}$$

By writing $w_1 = -W_{23}, w_2 = W_{13}, w_3 = -W_{12}$ this becomes

$$\begin{aligned}\mathbf{Wv} &= (w_2v_3 - w_3v_2)\mathbf{e}_1 + (w_3v_1 - w_1v_3)\mathbf{e}_2 + (w_1v_2 - w_2v_1)\mathbf{e}_3 \\ &\equiv \mathbf{w} \times \mathbf{v}.\end{aligned}$$

2.6 Scalar product of a CT(2)

The vector space CT(2) has a natural *scalar product*:

$$\mathbf{S} \cdot \mathbf{T} \equiv \mathrm{tr}(\mathbf{S}^T\mathbf{T}) = \mathrm{tr}(\mathbf{S}\mathbf{T}^T) = S_{ij}T_{ij}.$$

It is straightforward to prove that

(a) if $\mathbf{S}$ is symmetric,

$$\mathbf{S} \cdot \mathbf{T} = \mathbf{S} \cdot \mathbf{T}^T = \mathbf{S} \cdot \left[\frac{1}{2}(\mathbf{T} + \mathbf{T}^T)\right];$$

(b) if $\mathbf{W}$ is skew-symmetric,

$$\mathbf{W} \cdot \mathbf{T} = -\mathbf{W} \cdot \mathbf{T}^T = \mathbf{W} \cdot \left[\frac{1}{2}(\mathbf{T} - \mathbf{T}^T)\right];$$

(c) if $\mathbf{S}$ is symmetric and $\mathbf{W}$ is skew-symmetric,

$$\mathbf{S} \cdot \mathbf{T} = 0;$$

(d) if $\mathbf{T} \cdot \mathbf{S} = 0$ for every $\mathbf{S} \in$CT(2), then $\mathbf{T} = \mathbf{O}$;

(e) if $\mathbf{T} \cdot \mathbf{S} = 0$ for every symmetric tensor $\mathbf{S} \in$CT(2), then $\mathbf{T}$ is skew-symmetric;

(f) if $\mathbf{T} \cdot \mathbf{S} = 0$ for every tensor skew-symmetric $\mathbf{S} \in$CT(2), then $\mathbf{T}$ is symmetric.

A second-order Cartesian tensor $\mathbf{T}$ is *invertible* if there exists a tensor $\mathbf{T}^{-1}$, called the *inverse* of $\mathbf{T}$, such that

$$\mathbf{T}\mathbf{T}^{-1} = \mathbf{I} = \mathbf{T}^{-1}\mathbf{T}.$$

It can be proved that $\mathbf{T}$ is invertible if and only if $\det \mathbf{T} \neq 0$.

The identities

$$\begin{aligned}
\det(\mathbf{ST}) &= \det \mathbf{S} \det \mathbf{T}, \\
\det \mathbf{T}^T &= \det \mathbf{T}, \\
\det(\mathbf{T}^{-1}) &= (\det \mathbf{T})^{-1}, \\
(\mathbf{ST})^{-1} &= \mathbf{T}^{-1}\mathbf{S}^{-1}, \\
(\mathbf{T}^{-1})^T &= (\mathbf{T}^T)^{-1}
\end{aligned}$$

will be useful. For convenience, we use the abbreviation

$$\mathbf{S}^{-T} \equiv (\mathbf{S}^{-1})^T.$$

A tensor $\mathbf{Q}$ $\in$CT(2) is *orthogonal* if it preserves the scalar products:

$$\mathbf{Qu} \cdot \mathbf{Qv} = \mathbf{u} \cdot \mathbf{v} \quad \forall \mathbf{u}, \mathbf{v} \in \mathscr{V}. \tag{2.20}$$

The following proposition gives a characterization of orthogonal tensors.

Proposition 1. *A necessary and sufficient condition that* $\mathbf{Q}$ *be orthogonal is that*

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I}.$$

A second-order tensor $\mathbf{T}$ is *positive definite* provided

$$\mathbf{u} \cdot \mathbf{Tu} > 0 \quad \forall \mathbf{u} \in \mathscr{V} - \{\mathbf{0}\}.$$

Throughout this course we will use the following notation

$$\begin{aligned}
&\mathrm{Lin}^+ = \{\mathbf{T} \in \mathrm{CT}(2) : \det \mathbf{T} > 0\}, \\
&\mathrm{Sym} = \{\mathbf{T} \in \mathrm{CT}(2) : \mathbf{T}^T = \mathbf{T}\}, \\
&\mathrm{Skw} = \{\mathbf{T} \in \mathrm{CT}(2) : \mathbf{T}^T = -\mathbf{T}\}, \\
&\mathrm{Psym} = \{\mathbf{T} \in \mathrm{Sym} : \mathbf{T} \text{ positive definite}\}, \\
&\mathrm{Orth} = \{\mathbf{T} \in \mathrm{CT}(2) : \mathbf{T} \text{ orthogonal}\}, \\
&\mathrm{Orth}^+ = \{\mathbf{T} \in \mathrm{Orth} : \det \mathbf{T} = 1\}.
\end{aligned}$$

2.7 Contraction of tensors

Let $\mathbf{T}$ be a CT(n) with components $T_{ijk\ldots pq\ldots}$ with respect to the basis $\{\mathbf{e}_i\}$. Set $q = k$ and sum over k. These indices are then said to be *contracted*, and the effect of this operation is to produce a CT($n-2$). This procedure is illustrated by the following examples.

Examples

1. Suppose T_{ij} are the components of a CT(2), so that, under a transformation of basis, they change according to

$$T'_{ij} = l_{ip} l_{jq} T_{pq}.$$

Contraction of i with j gives

$$T'_{ii} = l_{ip}l_{iq}T_{pq} = \delta_{pq}T_{pq} = T_{pp} = T_{ii}.$$

Thus, T_{ii} is a CT(0). In fact, it is an example of a *scalar invariant* of $\mathbf{T}$, called the *trace* of $\mathbf{T}$, denoted $\mathrm{tr}(\mathbf{T})$.

2. Under contraction the tensor product $\mathbf{u} \otimes \mathbf{v}$ becomes the scalar product $\mathbf{u} \cdot \mathbf{v}$ since the components $u_i v_j$ contract to $u_i v_i$.

3. If T_{ijkl} are the components of a CT(4) then

$$T'_{ijkl} = l_{ip}l_{jq}l_{kr}l_{ls}T_{pqrs}.$$

Contraction of the indices k and l gives

$$T'_{ijkk} = l_{ip}l_{jq}\delta_{rs}T_{pqrs} = l_{ip}l_{jq}T_{pqrr},$$

so that T_{ijkk} are the components of a CT(2). [The proof in the general case is similar.]

2.8 The determinant of a CT(2)

Recall from Section 0 that if $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathscr{V}$ then their triple scalar product may be written as a determinant

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}$$

or, equivalently, in terms of the alternating symbol, as

$$\epsilon_{pqr}u_p v_q w_r = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}.$$

It follows that if T_{ij} are the components of a CT(2) then

$$\epsilon_{pqr}T_{pi}T_{qj}T_{rk} = \begin{vmatrix} T_{1i} & T_{1j} & T_{1k} \\ T_{2i} & T_{2j} & T_{2k} \\ T_{3i} & T_{3j} & T_{3k} \end{vmatrix}.$$

By considering different values of i, j, k we may deduce that

$$\begin{vmatrix} T_{1i} & T_{1j} & T_{1k} \\ T_{2i} & T_{2j} & T_{2k} \\ T_{3i} & T_{3j} & T_{3k} \end{vmatrix} = [\det(T_{mn})]\epsilon_{ijk},$$

where

$$\det(T_{mn}) = \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix}.$$

Hence

$$\epsilon_{pqr}T_{pi}T_{qj}T_{rk} = [\det(T_{mn})]\epsilon_{ijk}. \tag{2.21}$$

In particular, by taking $i = 1, j = 2, k = 3$, we obtain an expression for $\det(T_{mn})$, namely

$$\det(T_{mn}) = \epsilon_{pqr}T_{p1}T_{q2}T_{r3}. \tag{2.22}$$

An alternative expression for $\det(T_{mn})$ can be obtained by multiplying (2.20) by ϵ_{ijk} and noting that

$$\epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}, \quad \epsilon_{ijk}\epsilon_{ijk} = 6,$$

to give

$$\det(T_{mn}) = \frac{1}{6}\epsilon_{ijk}\epsilon_{pqr}T_{pi}T_{qj}T_{rk}. \tag{2.23}$$

Important example

The result (2.21) applies for any matrix (T_{ij}). In particular, application to (l_{ij}) and use of the fact that (l_{ij}) is proper orthogonal, so that $\det(l_{ij}) = 1$, leads to

$$\epsilon_{ijk} = \epsilon_{pqr}l_{pi}l_{qj}l_{rk}. \tag{2.24}$$

This indicates that ϵ_{ijk} are the components of a third-order *isotropic* tensor — see Section 2.9 for further discussion of isotropic tensors.

Example

We show that $\det(T_{ij})$ is independent of the choice of (orthonormal) basis.

$$\begin{aligned}
\det(T'_{ij}) &= \epsilon'_{pqr}T'_{p1}T'_{q2}T'_{r3} \qquad \text{using (2.22)} \\
&= \epsilon_{pqr}(l_{pi}l_{1j}T_{ij})(l_{qk}l_{2l}T_{kl})(l_{rm}l_{3n}T_{mn}) \\
&\qquad \text{using (2.24) and (2.12)} \\
&= \epsilon_{pqr}l_{pi}l_{qk}l_{rm}l_{1j}l_{2l}l_{3n}T_{ij}T_{kl}T_{mn} \\
&= \epsilon_{ikm}T_{ij}T_{kl}T_{mn}l_{1j}l_{2l}l_{3n} \\
&\qquad \text{using (2.24) and rearranging} \\
&= [\det(T_{ij})]\epsilon_{jln}l_{1j}l_{2l}l_{3n} \qquad \text{using (2.21)} \\
&= \det(T_{ij})\epsilon_{123} \qquad \text{(using (2.24))} \\
&= \det(T_{ij}).
\end{aligned}$$

This allows us to define the *determinant*, $\det \mathbf{T}$, of a CT(2) as $\det(T_{ij})$ for any choice of (orthonormal) basis. It is a *scalar invariant* of $\mathbf{T}$.

2.9 Isotropic tensors

Definition: if the components of a CT(n) $\mathbf{T}$ are unchanged under arbitrary *rotations* of orthonormal bases then $\mathbf{T}$ is said to be *isotropic*.

Examples

1. CT(0): all scalars are isotropic.

2. CT(1): $\mathbf{0}$ is the only isotropic vector.

This may be proved as follows. To satisfy the definition of isotropy we require

$$l_{ij}u_j \equiv u_i' = u_i \qquad \text{for all rotations } (l_{ij}).$$

We are free to choose l_{ij} to be any rotation. The choice

$$l_{ij} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(representing a rotation of $\pi/2$ about $\mathbf{e}_3$) leads to $u_1 = u_2 = 0$. Any other choice leads immediately to $u_3 = 0$ and hence $\mathbf{u} = \mathbf{0}$.

3. CT(2): scalar multiples of δ_{ij} are the only isotropic CT(2).

For isotropy we require

$$l_{ip}l_{jq}u_{pq} \equiv u_{ij}' = u_{ij} \qquad \text{for all rotations } (l_{ij}).$$

The same choice as in Example 2 gives $u_{11} = u_{22}, u_{13} = u_{23} = 0$. Choosing (l_{ij}) to represent a rotation through $\pi/2$ about $\mathbf{e}_1$ instead of $\mathbf{e}_3$ gives $u_{22} = u_{33}, u_{21} = u_{31} = 0$ etc. and we finish up with $u_{ij} = u_{11}\delta_{ij}$, where u_{11} is an arbitrary scalar.

4. CT(3): scalar multiples of ϵ_{ijk} are the only isotropic CT(3).

This can be proved in a similar way to the proof given in Example 2 (details are omitted since they are a bit tedious).

5. CT(4): the only independent isotropic CT(4)s with free indices i, j, k, l are scalar multiples of

$$\delta_{ij}\delta_{kl}, \quad \delta_{ik}\delta_{jl}, \quad \delta_{il}\delta_{jk}$$

and the most general isotropic CT(4) has components of the form

$$\alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk},$$

where α, β, γ are arbitrary scalars (the proof follows the same pattern as before and is omitted).

6. We use the result of Example 5 to prove the identity

$$\epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}.$$

Since ϵ_{ijk} is isotropic CT(3) it follows that $\epsilon_{ijk}\epsilon_{pqr}$ is isotropic CT(6) and hence, on contraction, we deduce that $\epsilon_{ijk}\epsilon_{pqk}$ is isotropic CT(4). But, the most general isotropic CT(4) has the form given in Example 5, so we can write

$$\epsilon_{ijk}\epsilon_{pqk} = \alpha\delta_{ij}\delta_{pq} + \beta\delta_{ip}\delta_{jq} + \gamma\delta_{iq}\delta_{jp}.$$

Setting $ij = 11, pq = 22$ in the above leads to $\alpha = 0$. Next, setting $ij = pq = 11$ gives $\beta + \gamma = 0$ and finally $ij = pq = 12$ gives $\beta = 1$. Hence the identity is established.

7. For tensors of higher order than 4 we note here that if n is even then the only isotropic CT(n)s are linear combinations of products of Kronecker deltas, while if n is odd the only isotropic CT(n)s are linear combinations of the products of one alternating symbol and an appropriate number of Kronecker deltas.

2.10 Eigenvalues and eigenvectors of a second-order tensor

Let $\mathbf{T}$ be a second-order tensor. A scalar λ is an *eigenvalue* of $\mathbf{T}$ if there exists a non-zero vector $\mathbf{u} \in \mathscr{V}$ such that

$$\mathbf{T}\mathbf{u} = \lambda\mathbf{u}, \tag{2.25}$$

in which case $\mathbf{u}$ is called an *eigenvector* of $\mathbf{T}$ corresponding to the eigenvalue λ.

The set of homogeneous equations (2.25) has non-trivial solutions if and only if

$$\det(\mathbf{T} - \lambda\mathbf{I}) = 0. \tag{2.26}$$

This is called the *characteristic equation* for $\mathbf{T}$, which is a third-degree algebraic equation in λ. Therefore any second-order tensor has at least one *real* eigenvalue and at most three *real* eigenvalues. The list of the eigenvalues of $\mathbf{T}$ in which each eigenvalue is repeated a number of times equal to its algebraic multiplicity is called the *spectrum* of $\mathbf{T}$.

Proposition 2. *The eigenvalues of a positive definite second-order tensor are strictly positive.*

Proof. Let λ be an eigenvalue of a positive definite tensor $\mathbf{T}$, and let $\mathbf{u}$ be a corresponding eigenvector, *i.e.* $\mathbf{T}\mathbf{u} = \lambda\mathbf{u}$. Then, since $\mathbf{u} \neq \mathbf{0}$, $\lambda|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{T}\mathbf{u} > 0$ and thus $\lambda > 0$. $\square$

Expansion of the determinant (2.26) leads to the equation

$$\lambda^3 - I_1(\mathbf{T})\lambda^2 + I_2(\mathbf{T})\lambda - I_3(\mathbf{T}) = 0, \tag{2.27}$$

where

$$\left.\begin{aligned} I_1(\mathbf{T}) &= \mathrm{tr}\mathbf{T}, \\ I_2(\mathbf{T}) &= \frac{1}{2}\left[(\mathrm{tr}\mathbf{T})^2 - \mathrm{tr}\mathbf{T}^2\right], \\ I_3(\mathbf{T}) &= \det\mathbf{T}. \end{aligned}\right\} \tag{2.28}$$

These quantities are called the *principal invariants* of $\mathbf{T}$. This terminology follows from the invariance of the trace and the determinant of a second-order tensor under change of basis of $\mathscr{V}$.

Cayley-Hamilton theorem: *Every second-order tensor* $\mathbf{T}$ *satisfies its own characteristic equation*

$$\mathbf{T}^3 - I_1(\mathbf{T})\mathbf{T}^2 + I_2(\mathbf{T})\mathbf{T} - I_3(\mathbf{T})\mathbf{I} = \mathbf{O}. \tag{2.29}$$

2.11 Symmetric second-order tensors

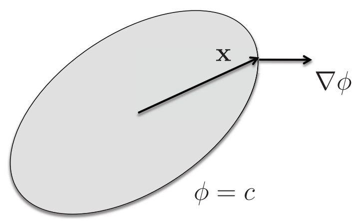
For symmetric second-order tensors the results of the previous section can be made more specific. Let $\mathbf{T}$ be a *symmetric* CT(2). Consider the quadratic form

$$\begin{aligned} \phi(\mathbf{x}) &\equiv (\mathbf{T}\mathbf{x})\cdot\mathbf{x} \equiv T_{ij}x_jx_i \\ &\equiv T_{11}x_1^2 + \cdots + 2T_{12}x_1x_2 + \ldots \end{aligned} \tag{2.30}$$

The equation

$$\phi(\mathbf{x}) = c, \tag{2.31}$$

where c is a constant, defines a quadratic surface (ellipsoid, paraboloid or hyperboloid). For example, if $T_{11} = 1/a^2$, $T_{22} = 1/b^2$, $T_{33} = 1/c^2$, $T_{12} = T_{23} = T_{13} = 0$ and the c in (2.31) is 1, then we have an ellipsoid with semi-axes a, b, c. The vector $\nabla\phi(\mathbf{x})$ is normal



to the surface (2.31) at the point $\mathbf{x}$ and from (2.30) we calculate

$$\nabla\phi(\mathbf{x}) = 2\mathbf{T}\mathbf{x}.$$

When the position vector $\mathbf{x}$ of a point P on the surface (2.31) is parallel to the normal at P then the direction OP is said to be a *principal direction* (or principal axis) of the quadratic surface, and hence a principal direction (or axis) or **eigenvector** of $\mathbf{T}$. This requires that $\nabla\phi(\mathbf{x})$ is proportional to $\mathbf{x}$, and hence

$$\mathbf{T}\mathbf{x} = \lambda\mathbf{x} \quad (T_{ij}x_j = \lambda x_i \text{ in index notation}), \tag{2.32}$$

where λ is a scalar—the *principal value* (or **eigenvalue**) associated with the *principal direction* (or **eigenvector**) $\mathbf{x}$.

Theorem

(a) There exist three mutually orthogonal eigenvectors, $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}$ say,

(b) the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are real,

(c) orthonormal basis vectors $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$, coincident with $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}$ can be chosen so that $\mathbf{T}$ has components $T'_{ij} = \lambda_i \delta_{ij}$ (no summation), i.e.

$$\mathbf{T} = \sum_{i=1}^{3} \lambda_i \mathbf{e}'_i \otimes \mathbf{e}'_i. \tag{2.33}$$

(c) is also known as the Spectral Representation.

Proof: from (2.32) we have

$$(\mathbf{T} - \lambda_i \mathbf{I})\mathbf{x}^{(i)} = \mathbf{0} \quad i \in \{1, 2, 3\}.$$

Choose axes $\mathbf{e}'_i$ coincident with $\mathbf{x}^{(i)}$, so that

$$\mathbf{T}\mathbf{e}'_i = \lambda_i \mathbf{e}'_i \quad \text{(no sum over } i\text{)}.$$

Next, take the dot product of this equation with $\mathbf{e}'_j$:

$$T'_{ij} \equiv T'_{ji} = \mathbf{e}'_j \cdot (\mathbf{T}\mathbf{e}'_i) = \lambda_i \mathbf{e}'_j \cdot \mathbf{e}'_i = \lambda_i \delta_{ij}.$$

Thus,

$$\begin{aligned} \mathbf{T} &= T'_{ij} \mathbf{e}'_i \otimes \mathbf{e}'_j \\ &= \lambda_i \delta_{ij} \mathbf{e}'_i \otimes \mathbf{e}'_j \quad \text{(sum over } i \text{ and } j\text{)} \\ &= \textstyle\sum_{i=1}^{3} t_i \mathbf{e}'_i \otimes \mathbf{e}'_i. \end{aligned}$$

2.12 Higher-order tensors

Extended tensor product notation A CT(n) can be represented in the form

$$\mathbf{T} = T_{ijk\ldots} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \ldots$$

The tensor product of a CT(m), $\mathbf{S}$, and a CT(n), $\mathbf{T}$, is a CT$(m+n)$ defined by

$$\begin{aligned} \mathbf{S} \otimes \mathbf{T} &\equiv (S_{ijk\ldots} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \ldots) \otimes (T_{pqr\ldots} \mathbf{e}_p \otimes \mathbf{e}_q \otimes \ldots) \\ &= S_{ijk\ldots} T_{pqr\ldots} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \cdots \otimes \mathbf{e}_p \otimes \mathbf{e}_q \otimes \ldots \end{aligned}$$

Note: there is no need to distinguish between, for example, $(\mathbf{e}_i \otimes \mathbf{e}_j) \otimes \mathbf{e}_k$ and $\mathbf{e}_i \otimes (\mathbf{e}_j \otimes \mathbf{e}_k)$, and we therefore write $\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$, omitting the brackets.

Note: if $\mathbf{S}$ and $\mathbf{T}$ are both CT(n)s and α, β are real numbers then $\alpha\mathbf{S} + \beta\mathbf{T}$ is also CT(n). CT(n)s therefore form a vector space of dimension 3^n with basis 'vectors' $\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \ldots$

3 Kinematics

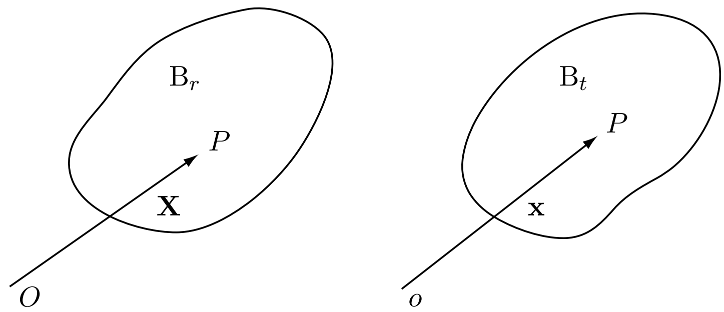
3.1 Bodies, configurations and motions

Definition: A *body* $\mathcal{B}$ is a set whose elements can be put into one-to-one correspondence with points of a region B in three-dimensional Euclidean point space.

The elements of $\mathcal{B}$ are called *particles* (or *material points*), and B is called a *configuration* of $\mathcal{B}$.

As the body moves the configuration changes with time. Let $t \in I \subset \mathbb{R}$ denote time, where I is an interval in $\mathbb{R}$. If, with each $t \in I$, we associate a unique configuration B_t of $\mathcal{B}$ then the family of configurations $\{\mathrm{B}_t : t \in I\}$ is called a *motion* of $\mathcal{B}$. We assume that as $\mathcal{B}$ moves continuously then B_t changes continuously.

It is convenient to identify a *reference configuration*, B_r say, which is an arbitrarily chosen fixed configuration. Then, any particle P of $\mathcal{B}$ may be labelled by its position vector $\mathbf{X}$ in B_r relative to some origin O. Let $\mathbf{x}$ be the position vector of P in the configuration B_t at time t relative to an origin o (which need not coincide with O).



We say that $\mathcal{B}$ *occupies* the configuration B_t at time t — B_t is also referred to as the *current configuration.*

[Note that B_r need not be a configuration actually occupied by $\mathcal{B}$ during the motion, but is often chosen to be the configuration occupied by $\mathcal{B}$ at some prescribed time.]

Since B_r and B_t are configurations of $\mathcal{B}$ there exists a bijection mapping $\boldsymbol{\chi} : \mathrm{B}_r \to \mathrm{B}_t$ such that

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) \qquad \text{for all } \mathbf{X} \in \mathrm{B}_r,$$
$$\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in \mathrm{B}_t. \ (3.1)$$

The mapping $\boldsymbol{\chi}$ is called the *deformation* of the body *from* B_r *to* B_t.

Since B_t depends on t we write

$$\mathbf{x} = \boldsymbol{\chi}_t(\mathbf{X}), \quad \mathbf{X} = \boldsymbol{\chi}_t^{-1}(\mathbf{x}) \tag{3.2}$$

instead of (3.1), or

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) \quad \text{for all } \mathbf{X} \in \mathrm{B}_r,\ t \in I. \tag{3.3}$$

For each particle P (with label $\mathbf{X}$) this describes the motion of P with t as parameter, and hence the motion of $\mathcal{B}$. It is usual to assume that $\boldsymbol{\chi}(\mathbf{X}, t)$ is twice-continuously differentiable with respect to position and time.

Example: *Rigid motion*

A motion is said to be it rigid if the distance between any two particles of $\mathcal{B}$ is invariant.

The motion defined by

$$\mathbf{x} \equiv \boldsymbol{\chi}(\mathbf{X}, t) = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{X}, \tag{3.4}$$

where $\mathbf{c}(t)$ is a vector and $\mathbf{Q}(t)$ is a proper orthogonal CT(2), is a rigid motion.

To show this we consider $\mathbf{Y} \in \mathrm{B}_r$ so that

$$\mathbf{y} = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{Y}.$$

Then

$$\begin{aligned} |\mathbf{x} - \mathbf{y}|^2 &= (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \\ &= [\mathbf{Q}(\mathbf{X} - \mathbf{Y})] \cdot [\mathbf{Q}(\mathbf{X} - \mathbf{Y})] \\ &= [\mathbf{Q}^T\mathbf{Q}(\mathbf{X} - \mathbf{Y})] \cdot (\mathbf{X} - \mathbf{Y}) \quad \text{(using (1.17))} \\ &= (\mathbf{X} - \mathbf{Y}) \cdot (\mathbf{X} - \mathbf{Y}) \quad (\text{since } \mathbf{Q}^T\mathbf{Q} = \mathbf{I}) \\ &= |\mathbf{X} - \mathbf{Y}|^2. \end{aligned}$$

In fact, although we have not proved it, *every* rigid motion can be expressed in the form (3.4). Note that $\mathbf{c}(t)$ represents a *translation* and $\mathbf{Q}(t)$ a *rotation*.

In the development of the basic principles of continuum mechanics a body $\mathcal{B}$ is endowed with various physical properties which are represented by scalar, vector and tensor fields defined on *either* B_r *or* B_t (for example, density, temperature, shape of surface). In the case of B_r the position vector $\mathbf{X}$ and time t serve as independent variables, and the fields are then said to be defined in terms of the *referential* or *material* description. Alternatively, in the case of B_t, $\mathbf{x}$ and t are used and the description is said to be *spatial.* The terminologies *Lagrangian* and *Eulerian descriptions* are also used in respect of B_r and B_t respectively.

Rectangular Cartesian coordinate systems with basis vectors $\{\mathbf{E}_i\}$ and $\{\mathbf{e}_i\}$ are chosen for B_r and B_t respectively, with *material coordinates* X_i and *spatial coordinates* x_i. Thus, relative to origins O and o respectively, we have

$$\mathbf{X} = X_i\mathbf{E}_i, \quad \mathbf{x} = x_i\mathbf{e}_i. \tag{3.5}$$

[Note that other vectors may be referred to either basis and tensors may be referred to either or to both simultaneously. Thus, a vector field $\mathbf{u}$ and a CT(2) $\mathbf{T}$ may be written

$$\mathbf{u} = u_i\mathbf{E}_i = \hat{u}_i\mathbf{e}_i,$$

$$\mathbf{T} = T_{ij}\mathbf{E}_i \otimes \mathbf{E}_j = \tilde{T}_{ij}\mathbf{E}_i \otimes \mathbf{e}_j = \bar{T}_{ij}\mathbf{e}_i \otimes \mathbf{E}_j = \hat{T}_{ij}\mathbf{e}_i \otimes \mathbf{e}_j,$$

for example.]

3.2 The material derivative

The *velocity* $\mathbf{v}$ of a particle P is defined as

$$\mathbf{v} \equiv \dot{\mathbf{x}} = \frac{\partial}{\partial t}\boldsymbol{\chi}(\mathbf{X}, t), \tag{3.6}$$

i.e. the rate of change of position of P (or $\partial/\partial t$ at fixed $\mathbf{X}$). The *acceleration* $\mathbf{a}$ of P is

$$\mathbf{a} \equiv \dot{\mathbf{v}} \equiv \ddot{\mathbf{x}} = \frac{\partial^2}{\partial t^2}\boldsymbol{\chi}(\mathbf{X}, t), \tag{3.7}$$

where the dot indicates differentiation with respect to t at fixed $\mathbf{X}$.

Let ϕ be a scalar field defined on B_t, i.e. $\phi(\mathbf{x}, t)$. Since $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, we may write

$$\phi(\mathbf{x}, t) = \phi[\boldsymbol{\chi}(\mathbf{X}, t), t] \equiv \Phi(\mathbf{X}, t), \tag{3.8}$$

which defines the notation Φ. Thus, any field defined on B_t (respectively B_r) can, through (3.2), equally be defined on B_r (respectively B_t).

The *material derivative* of ϕ is the rate of change of ϕ at fixed *material point* P, i.e. at fixed $\mathbf{X}$. We write the material derivative as $\dot{\phi}$ or $D\phi/Dt$.

By definition

$$\dot{\phi} = \frac{\partial}{\partial t}\Phi(\mathbf{X}, t).$$

By the chain rule for partial derivatives we then have

$$\begin{aligned} \tfrac{\partial}{\partial t}\Phi(\mathbf{X}, t) &= \tfrac{\partial}{\partial t}\phi(\mathbf{x}, t) + \tfrac{\partial x_i}{\partial t}\tfrac{\partial}{\partial x_i}\phi(\mathbf{x}, t) \\ &= \tfrac{\partial}{\partial t}\phi(\mathbf{x}, t) + \tfrac{\partial \mathbf{x}}{\partial t} \cdot \nabla\phi(\mathbf{x}, t). \end{aligned}$$

Using (3.6) we thus have

$$\frac{\partial}{\partial t}\Phi(\mathbf{X}, t) \equiv \dot{\phi} \equiv \frac{D\phi}{Dt} = \frac{\partial}{\partial t}\phi + \mathbf{v} \cdot \nabla\phi. \tag{3.9}$$

Similarly, for a vector field

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}[\boldsymbol{\chi}(\mathbf{X}, t), t] = \mathbf{U}(\mathbf{X}, t) \tag{3.10}$$

(which defines $\mathbf{U}$), we obtain

$$\frac{\partial}{\partial t}\mathbf{U}(\mathbf{X}, t) \equiv \dot{\mathbf{u}} \equiv \frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{u}. \tag{3.11}$$

In particular, the acceleration $\mathbf{a} = \dot{\mathbf{v}}$ is given by

$$\mathbf{a} = \dot{\mathbf{v}} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}. \tag{3.12}$$

We emphasize that

$$\left.\frac{\partial}{\partial t}\right|_{\mathbf{X}} = \frac{D}{Dt} \equiv \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} + \mathbf{v} \cdot \nabla. \tag{3.13}$$

3.3 Differentiation of Cartesian tensor fields

Let $\phi, \mathbf{u}, \mathbf{T}$ be scalar, vector and tensor functions of position $\mathbf{x}$. The operation of the gradient operator, grad or ∇, on these functions with respect to the basis $\{\mathbf{e}_i\}$ is defined as follows:

$$\operatorname{grad}\phi \equiv \nabla\phi = \frac{\partial\phi}{\partial x_i}\mathbf{e}_i,$$

$$\begin{aligned}\operatorname{grad}\mathbf{u} \equiv \nabla\otimes\mathbf{u} &= \tfrac{\partial\mathbf{u}}{\partial x_q}\otimes\mathbf{e}_q \\ &= \tfrac{\partial}{\partial x_q}(u_p\mathbf{e}_p)\otimes\mathbf{e}_q \\ &= \tfrac{\partial u_p}{\partial x_q}\mathbf{e}_p\otimes\mathbf{e}_q, \qquad (3.14)\end{aligned}$$

$$\begin{aligned}\operatorname{grad}\mathbf{T} \equiv \nabla\otimes\mathbf{T} &= \tfrac{\partial}{\partial x_i}\mathbf{T}\otimes\mathbf{e}_i \\ &= \tfrac{\partial}{\partial x_i}(T_{pq}\mathbf{e}_p\otimes\mathbf{e}_q)\otimes\mathbf{e}_i \\ &= \tfrac{\partial T_{pq}}{\partial x_i}\mathbf{e}_p\otimes\mathbf{e}_q\otimes\mathbf{e}_i, \qquad (3.15)\end{aligned}$$

and similarly for higher-order tensors. Note that the operation of grad increases the order of the tensor by one. *Contraction* of $\operatorname{grad}\mathbf{u}$ gives $\nabla\cdot\mathbf{u}$. There are several possible contractions of $\nabla\otimes\mathbf{T}$. We *define* $\operatorname{div}\mathbf{T}$ as follows.

$$\operatorname{div}\mathbf{T} \equiv \frac{\partial T_{pq}}{\partial x_i}\mathbf{e}_q(\mathbf{e}_p\cdot\mathbf{e}_i)$$

— the p—i contraction. Since $\mathbf{e}_p\cdot\mathbf{e}_i = \delta_{ip}$ we obtain

$$\operatorname{div}\mathbf{T} = \frac{\partial T_{pq}}{\partial x_p}\mathbf{e}_q. \tag{3.16}$$

Exercise Show that (3.11) can be written as

$$\dot{\mathbf{u}} = \frac{\partial\mathbf{u}}{\partial t} + (\nabla\otimes\mathbf{u})\mathbf{v}. \tag{3.17}$$

3.4 Deformation gradient

Let Grad, Div, Curl (respectively grad, div, curl) denote the gradient, divergence and curl operators in the reference (respectively current) configuration, i.e. with respect to $\mathbf{X}$ (respectively $\mathbf{x}$).

We define the *deformation gradient tensor* $\mathbf{F}$ as

$$\mathbf{F}(\mathbf{X},t) = \operatorname{Grad}\mathbf{x} \equiv \operatorname{Grad}\boldsymbol{\chi}(\mathbf{X},t). \tag{3.18}$$

With respect to the chosen basis vectors and with use of (3.14) we have

$$\mathbf{F} = \frac{\partial}{\partial X_j}(x_i\mathbf{e}_i)\otimes\mathbf{E}_j = \frac{\partial x_i}{\partial X_j}\mathbf{e}_i\otimes\mathbf{E}_j$$

or, in component form,

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \tag{3.19}$$

with $x_i = \chi_i(\mathbf{X}, t)$.

We assume that $\det \mathbf{F} \neq 0$ (to be justified shortly) so that $\mathbf{F}$ has an inverse $\mathbf{F}^{-1}$, given by

$$\mathbf{F}^{-1} = \operatorname{grad} \mathbf{X}, \tag{3.20}$$

with components

$$(\mathbf{F}^{-1})_{ij} = \frac{\partial X_i}{\partial x_j}. \tag{3.21}$$

This may be checked by means of the calculation

$$(\mathbf{F}\mathbf{F}^{-1})_{ij} = F_{ik}(\mathbf{F}^{-1})_{kj} = \frac{\partial x_i}{\partial X_k}\frac{\partial X_k}{\partial x_j} = \frac{\partial x_i}{\partial x_j} = \delta_{ij}.$$

It follows from (3.19) that

$$F_{ij} dX_j = \frac{\partial x_i}{\partial X_j} dX_j = dx_i,$$

i.e.

$$d\mathbf{x} = \mathbf{F} d\mathbf{X}, \tag{3.22}$$

which has inverse

$$d\mathbf{X} = \mathbf{F}^{-1} d\mathbf{x}. \tag{3.23}$$

Equation (3.22) describes how small *line elements* $d\mathbf{X}$ of material at $\mathbf{X}$ transform under the deformation into $d\mathbf{x}$ (which consists of the same material as $d\mathbf{X}$) at $\mathbf{x}$.

Line elements transform *linearly* since $\mathbf{F}$ depends on $\mathbf{X}$ (and not on $d\mathbf{X}$). Thus, at each $\mathbf{X}$, $\mathbf{F}$ is a *linear mapping* (i.e. a second-order tensor).

If $\mathbf{F}$ is independent of $\mathbf{X}$ then the deformation is said to be *homogeneous* (the same at each point of the body).

We justify taking $\mathbf{F}$ to be *non-singular* ($\det \mathbf{F} \neq 0$) by noting that $\mathbf{F} d\mathbf{X} \neq \mathbf{0}$ if $d\mathbf{X} \neq \mathbf{0}$ — a line element cannot be annihilated.

Example

Let $\phi, \mathbf{u}, \mathbf{T}$ be CT(0), CT(1), CT(2) fields associated with a moving body. We establish the following formulas:

$$\begin{aligned}&\operatorname{Grad}\phi = \mathbf{F}^T \operatorname{grad}\phi, \quad \operatorname{Grad}\mathbf{u} = (\operatorname{grad}\mathbf{u})\mathbf{F},\\ &\operatorname{Div}\mathbf{u} = J\operatorname{div}(J^{-1}\mathbf{F}\mathbf{u}), \quad \operatorname{Div}\mathbf{T} = J\operatorname{div}(J^{-1}\mathbf{F}\mathbf{T}), \ (3.24)\end{aligned}$$

where J is defined as

$$J = \det \mathbf{F}. \tag{3.25}$$

First, we calculate

$$\begin{aligned}\mathbf{F}^T \operatorname{grad}\phi &= \left(\tfrac{\partial x_i}{\partial X_j}\mathbf{e}_i \otimes \mathbf{E}_j\right)^T \tfrac{\partial \phi}{\partial x_p}\mathbf{e}_p\\ &= \tfrac{\partial x_i}{\partial X_j}(\mathbf{E}_j \otimes \mathbf{e}_i)\mathbf{e}_p \tfrac{\partial \phi}{\partial x_p}\\ &= \tfrac{\partial x_i}{\partial X_j}\tfrac{\partial \phi}{\partial x_p}\mathbf{E}_j \delta_{ip}\\ &= \tfrac{\partial x_p}{\partial X_j}\tfrac{\partial \phi}{\partial x_p}\mathbf{E}_j = \tfrac{\partial \phi}{\partial X_j}\mathbf{E}_j = \operatorname{Grad}\phi.\end{aligned}$$

Next,

$$\begin{aligned}(\operatorname{grad}\mathbf{u})\mathbf{F} &= \left(\tfrac{\partial u_i}{\partial x_j}\mathbf{e}_i \otimes \mathbf{e}_j\right)\left(\tfrac{\partial x_p}{\partial X_q}\mathbf{e}_p \otimes \mathbf{E}_q\right)\\ &= \tfrac{\partial u_i}{\partial x_j}\tfrac{\partial x_p}{\partial X_q}(\mathbf{e}_i \otimes \mathbf{e}_j)(\mathbf{e}_p \otimes \mathbf{E}_q)\\ &= \tfrac{\partial u_i}{\partial x_j}\tfrac{\partial x_p}{\partial X_q}\mathbf{e}_i \otimes \mathbf{E}_q \delta_{jp}\\ &= \tfrac{\partial u_i}{\partial x_p}\tfrac{\partial x_p}{\partial X_q}\mathbf{e}_i \otimes \mathbf{E}_q\\ &= \tfrac{\partial u_i}{\partial X_q}\mathbf{e}_i \otimes \mathbf{E}_q = \operatorname{Grad}\mathbf{u}.\end{aligned}$$

For the right-hand side of the third equation in (3.24), we calculate

$$\begin{aligned}J\operatorname{div}(J^{-1}\mathbf{F}\mathbf{u}) &= J\tfrac{\partial}{\partial x_p}(J^{-1}F_{pq}u_q)\\ &= F_{pq}\tfrac{\partial u_q}{\partial x_p} + Ju_q \tfrac{\partial}{\partial x_p}(J^{-1}F_{pq}). \ (*)\end{aligned}$$

But,

$$\begin{aligned}\tfrac{\partial}{\partial x_p}(J^{-1}F_{pq}) &= \tfrac{\partial X_r}{\partial x_p}\tfrac{\partial}{\partial X_r}(J^{-1}F_{pq})\\ &= -J^{-2}\tfrac{\partial J}{\partial X_r}\tfrac{\partial X_r}{\partial x_p}F_{pq} + J^{-1}\tfrac{\partial X_r}{\partial x_p}\tfrac{\partial F_{pq}}{\partial X_r}\\ &= -J^{-2}\left(J(\mathbf{F}^{-1})_{ts}\tfrac{\partial F_{st}}{\partial X_r}\right)\underbrace{\frac{\partial X_r}{\partial x_p}\frac{\partial x_p}{\partial X_q}}_{\delta_{rq}} + J^{-1}\tfrac{\partial X_r}{\partial x_p}\tfrac{\partial^2 x_p}{\partial X_q \partial X_r}\end{aligned}$$

In the above, we have used $F_{pq} = \partial x_p / \partial X_q$ and the relation

$$\frac{\partial}{\partial t}(\det \mathbf{T}) = (\det \mathbf{T})\operatorname{tr}\left(\mathbf{T}^{-1}\frac{\partial \mathbf{T}}{\partial t}\right). \tag{3.26}$$

Thus,

$$\frac{\partial}{\partial x_p}(J^{-1}F_{pq}) = -J^{-1}\frac{\partial X_t}{\partial x_s}\frac{\partial^2 x_s}{\partial X_q \partial X_t} + J^{-1}\frac{\partial X_r}{\partial x_p}\frac{\partial^2 x_p}{\partial X_q \partial X_r} = 0.$$

Hence, $(*)$ gives

$$J\operatorname{div}(J^{-1}\mathbf{F}\mathbf{u}) = F_{pq}\frac{\partial u_q}{\partial x_p} = \frac{\partial x_p}{\partial X_q}\frac{\partial u_q}{\partial x_p} = \frac{\partial u_q}{\partial X_q} = \operatorname{Div}\mathbf{u}.$$

Similarly,

$$\begin{aligned} J\mathrm{div}\,(J^{-1}\mathbf{F}\mathbf{T}) &= J\tfrac{\partial}{\partial x_p}(J^{-1}F_{pq}T_{qr}\mathbf{E}_r) \\ &= \underbrace{J\frac{\partial}{\partial x_p}(J^{-1}F_{pq})T_{qr}\mathbf{E}_r}_{=0} + F_{pq}\tfrac{\partial T_{qr}}{\partial x_p}\mathbf{E}_r \\ &= \tfrac{\partial x_p}{\partial X_q}\tfrac{\partial T_{qr}}{\partial x_p}\mathbf{E}_r = \tfrac{\partial T_{qr}}{\partial X_q}\mathbf{E}_r = \mathrm{Div}\,\mathbf{T}. \end{aligned}$$

Proof of (3.26) For a small increment δ in time t we have

$$\mathbf{F}(\mathbf{X},\tau+\delta) = \mathbf{F}(\mathbf{X},\tau) + \delta\dot{\mathbf{F}}(\mathbf{X},\tau) + \mathcal{O}(\delta^2).$$

Then

$$\frac{\partial J}{\partial t} = \lim_{\delta\to 0}\frac{1}{\delta}\left[J(\mathbf{X},t+\delta) - J(\mathbf{X},t)\right] = \lim_{\delta\to 0}\frac{1}{\delta}\left[\det(\mathbf{F}+\delta\dot{\mathbf{F}}) - J\right].$$

But

$$\begin{aligned} \det(\mathbf{F}+\delta\dot{\mathbf{F}}) &= \det[\mathbf{F}(\mathbf{I}+\delta\mathbf{F}^{-1}\dot{\mathbf{F}})] = (\det\mathbf{F})\det(\mathbf{I}+\delta\mathbf{F}^{-1}\dot{\mathbf{F}}) \\ &= J\det[\delta(\mathbf{F}^{-1}\dot{\mathbf{F}}+\delta^{-1}\mathbf{I})] = J\delta^3\det(\mathbf{F}^{-1}\dot{\mathbf{F}}+\delta^{-1}\mathbf{I}) \\ &= \delta^3 J[\delta^{-3} + \delta^{-2}I_1(\mathbf{F}^{-1}\dot{\mathbf{F}}) + \delta^{-1}I_2(\mathbf{F}^{-1}\dot{\mathbf{F}}) + I_3(\mathbf{F}^{-1}\dot{\mathbf{F}})] \\ &= J[1+\delta\, I_1(\mathbf{F}^{-1}\dot{\mathbf{F}}) + \mathcal{O}(\delta^2)], \\ &= J[1+\delta\,\mathrm{tr}(\mathbf{F}^{-1}\dot{\mathbf{F}}) + \mathcal{O}(\delta^2)]. \end{aligned}$$

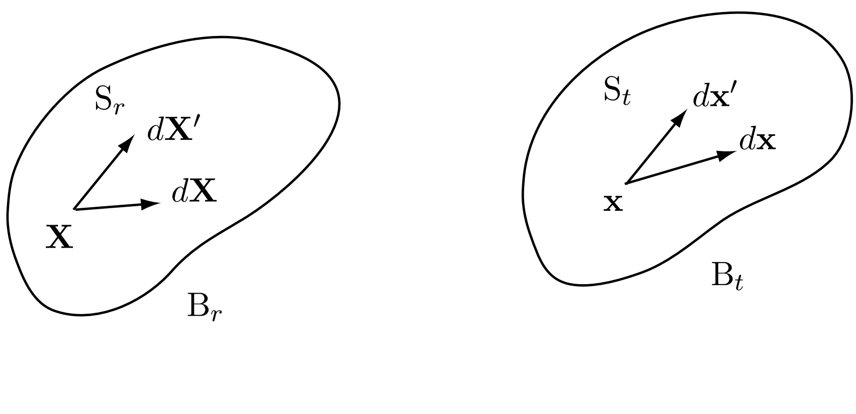
Now the limit is easily computed as (3.26).

By following similar arguments one can readily get the components of Grad$J$:

$$\frac{\partial}{\partial X_r}(\det\mathbf{F}) = (\det\mathbf{F})\,\mathrm{tr}\left(\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial X_r}\right).$$

For an alternative proof, see Q8 in Example Sheet 3.

3.5 Deformation of area and volume elements



Consider a surface S$_r$ in B$_r$ which deforms into the surface S$_t$ in B$_t$. Let $\mathbf{X}$ be a point on S$_r$ and $\mathbf{x}$ the corresponding point on S$_t$. Let $d\mathbf{X}$ and $d\mathbf{X}'$ be line elements of material on

S_r based at $\mathbf{X}$ and let $d\mathbf{x}$ and $d\mathbf{x}'$ be their images (on S_t) under the deformation. If $\mathbf{F}$ denotes the deformation gradient, then

$$d\mathbf{x} = \mathbf{F}d\mathbf{X}, \quad d\mathbf{x}' = \mathbf{F}d\mathbf{X}'. \tag{3.27}$$

Let dA and da be surface area elements on S_r and S_t respectively, and let $\mathbf{N}$ and $\mathbf{n}$ be unit normals at $\mathbf{X}$ and $\mathbf{x}$ respectively.

For the parallelogram with sides $d\mathbf{X}, d\mathbf{X}'$ we have

$$\mathbf{N}dA = d\mathbf{X} \times d\mathbf{X}'.$$

Under the deformation this becomes a parallelogram with sides $d\mathbf{x}, d\mathbf{x}'$ and area

$$\mathbf{n}da = d\mathbf{x} \times d\mathbf{x}'.$$

From (3.27) we obtain

$$\begin{aligned} \mathbf{F}^T\mathbf{n}da &= \mathbf{F}^T[(\mathbf{F}d\mathbf{X}) \times (\mathbf{F}d\mathbf{X}')] \\ &= (\det \mathbf{F})d\mathbf{X} \times d\mathbf{X}' \end{aligned}$$

using the result from question 10 of Problem Sheet 3. Hence

$$\mathbf{n}da = J(\mathbf{F}^T)^{-1}\mathbf{N}dA,$$

where $J = \det \mathbf{F}$. With the notation

$$\mathbf{F}^{-T} = (\mathbf{F}^T)^{-1} = (\mathbf{F}^{-1})^T,$$

this becomes

$$\mathbf{n}da = J\mathbf{F}^{-T}\mathbf{N}dA. \tag{3.28}$$

This is an important result — it is known as *Nanson's formula*, and it describes how elements of surface area deform.

Next, consider the parallelepiped in B_r formed by line elements $d\mathbf{X}, d\mathbf{X}', d\mathbf{X}''$ at $\mathbf{X}$. Its volume dV is given by

$$dV = d\mathbf{X} \cdot (d\mathbf{X}' \times d\mathbf{X}'') = \left| \, d\mathbf{X} \; d\mathbf{X}' \; d\mathbf{X}'' \, \right|.$$

The corresponding volume dv in B_t is

$$dv = d\mathbf{x} \cdot (d\mathbf{x}' \times d\mathbf{x}'') = \left| \, d\mathbf{x} \; d\mathbf{x}' \; d\mathbf{x}'' \, \right| = \left| \, \mathbf{F}d\mathbf{X} \; \mathbf{F}d\mathbf{X}' \; \mathbf{F}d\mathbf{X}'' \, \right| = |\mathbf{F}| \left| \, d\mathbf{X} \; d\mathbf{X}' \; d\mathbf{X}'' \, \right|,$$

i.e.

$$dv = JdV. \tag{3.29}$$

By convention, volume is taken to be positive, so that

$$J \equiv \det \mathbf{F} > 0. \tag{3.30}$$

From (3.29) we see that J is a measure of the change in volume under the deformation. If the deformation is such that there is no change in volume then the deformation is said to be *isochoric*, and then

$$J \equiv \det \mathbf{F} = 1. \tag{3.31}$$

For some materials many deformations are such that (3.31) holds to a good approximation, and (3.31) is adopted as an *idealization*. An (ideal) material for which (3.31) holds for *all* deformations is called an *incompressible material*.

Example

A (time-dependent) deformation is defined by

$$x_1 = \alpha X_1 + \beta X_2, \quad x_2 = \alpha^{-1} X_2, \quad x_3 = X_3$$

(the same basis vectors being chosen for $\mathbf{X}$ and $\mathbf{x}$), where $\alpha\,(\neq 0)$ and β are constants. Show that the deformation is isochoric.

With respect to the given Cartesian coordinates the deformation gradient $\mathbf{F}$ has components

$$(F_{ij}) \equiv \left(\frac{\partial x_i}{\partial X_j}\right) = \begin{bmatrix} \alpha & \beta & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Clearly, $\det \mathbf{F} = 1$, so the deformation is isochoric. It corresponds to stretching by a factor α in the x_1 direction, compressing by a factor α^{-1} in the x_2 direction and then shearing by an amount β parallel to the x_1 direction, as illustrated below by application to a square with sides of unit length.

3.6 Further results from tensor algebra

The square root theorem

If $\mathbf{S}$ is a positive definite, symmetric CT(2) then there exists a unique, positive definite, symmetric CT(2), $\mathbf{U}$ say, such that $\mathbf{U}^2 = \mathbf{S}$.

Proof Since $\mathbf{S}$ is symmetric we may write it in the spectral form

$$\mathbf{S} = \sum_{i=1}^{3} s_i \mathbf{e}'_i \otimes \mathbf{e}'_i,$$

where s_i are the (real) eigenvalues of $\mathbf{S}$ and $\{\mathbf{e}'_i\}$ are the (unit) eigenvectors. Since $\mathbf{S}$ is positive definite, we have $s_i > 0$. Now define $\mathbf{U}$ by

$$\mathbf{U} = \sum_{i=1}^{3} \sqrt{s_i}\, \mathbf{e}'_i \otimes \mathbf{e}'_i.$$

Then, $\mathbf{U}$ is positive definite and symmetric and $\mathbf{U}^2 = \mathbf{S}$, as required. Uniqueness is obvious – convince yourselves of this.

The polar decomposition theorem

Let $\mathbf{F}$ be a CT(2) such that $\det \mathbf{F} > 0$. Then there exist unique, positive definite, symmetric tensors, $\mathbf{U}$ and $\mathbf{V}$, and a unique proper orthogonal tensor $\mathbf{R}$ such that

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}. \tag{3.32}$$

Proof The tensors $\mathbf{F}\mathbf{F}^T$ and $\mathbf{F}^T\mathbf{F}$ are symmetric and positive definite. Hence, by the square root theorem, there exist unique positive definite symmetric tensors $\mathbf{U}$, $\mathbf{V}$ such that

$$\mathbf{V}^2 = \mathbf{F}\mathbf{F}^T, \quad \mathbf{U}^2 = \mathbf{F}^T\mathbf{F}.$$

Now define $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$. We need to prove that $\mathbf{R}$ is proper orthogonal. First, we calculate

$$\mathbf{R}^T\mathbf{R} = (\mathbf{F}\mathbf{U}^{-1})^T(\mathbf{F}\mathbf{U}^{-1}) = \mathbf{U}^{-1}\mathbf{F}^T\mathbf{F}\mathbf{U}^{-1} = \mathbf{U}^{-1}\mathbf{U}^2\mathbf{U}^{-1} = \mathbf{I},$$

and hence we deduce that $\mathbf{R}$ is orthogonal. Second, we calculate

$$\det \mathbf{R} = \det(\mathbf{F}\mathbf{U}^{-1}) = (\det \mathbf{F})(\det \mathbf{U})^{-1} > 0,$$

and it follows that $\mathbf{R}$ is *proper* orthogonal.

Since $\mathbf{U}$ is unique, $\mathbf{R}$ is unique and hence $\mathbf{F} = \mathbf{R}\mathbf{U}$. Similarly, $\mathbf{F} = \mathbf{V}\mathbf{S}$, where $\mathbf{S}$ is proper orthogonal. Thus,

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{S} = \mathbf{R}\mathbf{U}\mathbf{R}^T\mathbf{R}.$$

By uniqueness it follows that $\mathbf{S} = \mathbf{R}$ and hence (3.32) holds. Note that $\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^T$.

Corollary

If $\mathbf{U}$ has eigenvalues λ_i and eigenvectors $\mathbf{u}^{(i)}$, $i \in \{1, 2, 3\}$, then $\lambda_i > 0$ and λ_i are also the eigenvalues of $\mathbf{V}$ with eigenvectors $\mathbf{R}\mathbf{u}^{(i)}$.

Proof $\lambda_i > 0$ follows from symmetry and positive definiteness of $\mathbf{U}$. Also, we have

$$\mathbf{V}(\mathbf{R}\mathbf{u}^{(i)}) = \mathbf{V}\mathbf{R}\mathbf{u}^{(i)} = \mathbf{R}\mathbf{U}\mathbf{u}^{(i)} = \mathbf{R}(\lambda_i\mathbf{u}^{(i)}) = \lambda_i(\mathbf{R}\mathbf{u}^{(i)}),$$

which shows that $\mathbf{R}\mathbf{u}^{(i)}$ are the eigenvectors of $\mathbf{V}$.

3.7 Analysis of deformation

3.7.1 Stretch, extension, shear and strain

Let $\mathbf{M}$ and $\mathbf{m}$ be unit vectors along $d\mathbf{X}$ and $d\mathbf{x}$ respectively, so that $d\mathbf{X} = \mathbf{M}|d\mathbf{X}|, d\mathbf{x} = \mathbf{m}|d\mathbf{x}|$ and (3.22) gives

$$\mathbf{m}|d\mathbf{x}| = \mathbf{F}\mathbf{M}|d\mathbf{X}|.$$

Thus
$$|d\mathbf{x}|^2 = (\mathbf{FM}) \cdot (\mathbf{FM})|d\mathbf{X}|^2 = (\mathbf{F}^T\mathbf{FM}) \cdot \mathbf{M}|d\mathbf{X}|^2 \tag{3.33}$$
and hence
$$\frac{|d\mathbf{x}|}{|d\mathbf{X}|} = |\mathbf{FM}| = [\mathbf{M} \cdot (\mathbf{F}^T\mathbf{FM})]^{1/2} \equiv \lambda(\mathbf{M}), \tag{3.34}$$
which defines $\lambda(\mathbf{M})$ — the *stretch in the direction* $\mathbf{M}$ at $\mathbf{X}$.

Note that $0 < \lambda(\mathbf{M}) < \infty$.

Now consider a pair of line elements $d\mathbf{X}_1, d\mathbf{X}_2$ based at $\mathbf{X}$, so that
$$d\mathbf{x}_1 = \mathbf{F}d\mathbf{X}_1, \quad d\mathbf{x}_2 = \mathbf{F}d\mathbf{X}_2$$
and the angle between them is given by
$$\cos\Theta = \mathbf{M}_1 \cdot \mathbf{M}_2, \quad \cos\theta = \frac{\mathbf{M}_1 \cdot (\mathbf{F}^T\mathbf{F}\mathbf{M}_2)}{\lambda(\mathbf{M_1})\lambda(\mathbf{M_2})}$$
before and after deformation respectively.

The decrease in angle $\Theta - \theta$ (which may be positive or negative) is called the *shear* of the direction $\mathbf{M}_1, \mathbf{M}_2$ in the plane of $\mathbf{M}_1, \mathbf{M}_2$.

Next, from (3.33) we have
$$|d\mathbf{x}|^2 - |d\mathbf{X}|^2 = d\mathbf{X} \cdot (\mathbf{F}^T\mathbf{F} - \mathbf{I})d\mathbf{X}. \tag{3.35}$$
The material is said to be *unstrained* at $\mathbf{X}$ if no line element changes length, i.e.
$$d\mathbf{X} \cdot (\mathbf{F}^T\mathbf{F} - \mathbf{I})d\mathbf{X} = 0 \quad \text{for all } d\mathbf{X},$$
or, equivalently,
$$\lambda(\mathbf{M}) = 1 \quad \text{for all } \textit{unit} \text{ vectors } \mathbf{M}.$$
It follows that $\mathbf{F}^T\mathbf{F} - \mathbf{I} = \mathbf{O}$, which allows the possibility that $\mathbf{F}$ is just a rotation $\mathbf{R}$, since, for orthogonal $\mathbf{R}$, we have $\mathbf{R}^T\mathbf{R} = \mathbf{I}$.

Strain is measured locally by changes in the lengths of line elements, i.e. by the value of (3.35). Thus, the tensor $\mathbf{F}^T\mathbf{F} - \mathbf{I}$ is a measure of strain. The so-called *Green strain tensor* $\mathbf{E}$ is defined by
$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}). \tag{3.36}$$
Using the polar decomposition (3.32) for the deformation gradient $\mathbf{F}$, we may also form the following tensor measures of deformation:
$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2,$$
$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2. \ (3.37)$$

We refer to $\mathbf{C}$ and $\mathbf{B}$ as the *right* and *left Cauchy-Green deformation tensors* respectively.

Since $\mathbf{U}$ is positive definite and symmetric there exist (unit) eigenvectors $\mathbf{u}^{(i)}$ such that

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}, \tag{3.38}$$

where $\lambda_i > 0$ are the *principal stretches* of the deformation and $\mathbf{u}^{(i)}$ are the *principal directions*. Note that, in accordance with the definition (3.34), $\lambda_i = \lambda(\mathbf{u}^{(i)})$ — hence the terminology *principal stretch*.

$\mathbf{U}$ and $\mathbf{V}$ are called the *right* and *left stretch tensors* respectively. The deformation $\mathbf{F}$ rotates the principal axes of $\mathbf{U}$ into those of $\mathbf{V}$ as well as stretching along those directions. The principal axes of $\mathbf{U}$ and $\mathbf{V}$ are sometimes referred to as the *Lagrangian* and *Eulerian principal axes* respectively.

The *displacement* $\mathbf{u}$ of a particle is defined as

$$\mathbf{u} = \mathbf{x} - \mathbf{X},$$

so that

$$\mathbf{x} = \mathbf{X} + \mathbf{u}$$

and

$$\mathbf{F} = \operatorname{Grad} \mathbf{x} = \mathbf{I} + \operatorname{Grad} \mathbf{u}, \tag{3.39}$$

where $\operatorname{Grad} \mathbf{u}$ is the *displacement gradient* and $\mathbf{I}$ is the identity tensor.

[Note that $(\operatorname{Grad} \mathbf{X})_{ij} = \partial X_i / \partial X_j = \delta_{ij}$.]

3.7.2 Examples of deformations

1. Homogeneous deformation.

The most general form of homogeneous deformation is given by $\mathbf{x} = \mathbf{F}\mathbf{X} + \mathbf{c}$, with $\mathbf{F}$ independent of $\mathbf{X}$. The following examples are all special cases of this.

2. Homogeneous simple elongation of a circular cylinder (with lateral contraction).

$$\mathbf{F} = \mathbf{U} = \lambda_1 \mathbf{u}^{(1)} \otimes \mathbf{u}^{(1)} + \lambda_2 (\mathbf{u}^{(2)} \otimes \mathbf{u}^{(2)} + \mathbf{u}^{(3)} \otimes \mathbf{u}^{(3)}),$$

with $\mathbf{u}^{(1)}$ along the axis of the cylinder.

3. Homogeneous pure dilatation.

This is defined by $\lambda_1 = \lambda_2 = \lambda_3$, $\mathbf{F} = \lambda_1 \mathbf{I}$ and might be associated with, for example, the deformation of a cube into a cube of a different size.

4. Homogeneous pure shear.

This is defined by $\lambda_1 = \lambda, \lambda_2 = \lambda^{-1}, \lambda_3 = 1$, for example, or

$$\mathbf{F} = \lambda \mathbf{u}^{(1)} \otimes \mathbf{u}^{(1)} + \lambda^{-1}\mathbf{u}^{(2)} \otimes \mathbf{u}^{(2)} + \mathbf{u}^{(3)} \otimes \mathbf{u}^{(3)}.$$

5. Homogeneous simple shear. Simple shear is defined by the equations

$$x_1 = X_1 + \gamma X_2, \quad x_2 = X_2, \quad x_3 = X_3,$$

where γ (constant) is called the *amount of shear*, $\tan^{-1}\gamma$ is the shear of the directions $\mathbf{e}_1, \mathbf{e}_2$, and the same basis vectors are used for both reference and current coordinates.

The deformation gradient $\mathbf{F}$ has components

$$(F_{ij}) = \left(\frac{\partial x_i}{\partial X_j}\right) = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This is effectively a problem confined to the $(1,2)$-plane so we now restrict attention to this plane and write

$$(F_{ij}) = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}.$$

To find the Lagrangian principal axes we consider

$$\mathbf{U}^2 = \mathbf{F}^T\mathbf{F} = \begin{bmatrix} 1 & \gamma \\ \gamma & 1+\gamma^2 \end{bmatrix}. \tag{$*$}$$

The characteristic equation for $\mathbf{U}^2$, from which the eigenvalues λ^2 are determined, is

$$\det(\mathbf{U}^2 - \lambda^2\mathbf{I}) = 0,$$

i.e.

$$\begin{vmatrix} 1-\lambda^2 & \gamma \\ \gamma & 1-\lambda^2+\gamma^2 \end{vmatrix},$$

or, when expanded out,

$$\lambda^4 - (2+\gamma^2)\lambda^2 + 1 = 0.$$

Let the roots be λ_1^2, λ_2^2. Then

$$\lambda_1^2 + \lambda_2^2 = 2 + \gamma^2, \quad \lambda_1^2\lambda_2^2 = 1.$$

[Note that $\lambda_3 = 1$ corresponds to $\mathbf{u}^{(3)} = \mathbf{e}_3$.]

Now set $\lambda_1 = \lambda \geq 1$, $\lambda_2 = \lambda^{-1}$ so that

$$\lambda^2 + \lambda^{-2} = 2 + \gamma^2$$

and hence

$$\gamma = \lambda - \lambda^{-1}, \quad \lambda = \frac{1}{2}\gamma + \sqrt{1 + \frac{1}{4}\gamma^2}.$$

Let

$$\mathbf{u}^{(1)} = (\cos\theta, \sin\theta, 0), \quad \mathbf{u}^{(2)} = (-\sin\theta, \cos\theta, 0).$$

Then the representation

$$\mathbf{U}^2 = \lambda_1^2 \mathbf{u}^{(1)} \otimes \mathbf{u}^{(1)} + \lambda_2^2 \mathbf{u}^{(2)} \otimes \mathbf{u}^{(2)} + \lambda_3^2 \mathbf{u}^{(3)} \otimes \mathbf{u}^{(3)}$$

yields, when restricted to two dimensions,

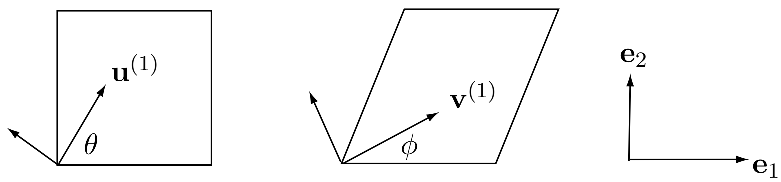
$$\mathbf{U}^2 = \lambda^2 \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} + \lambda^{-2} \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix}.$$

Comparison with $(*)$ shows that

$$\begin{aligned} \lambda^2\cos^2\theta + \lambda^{-2}\sin^2\theta &= 1, \\ \lambda^2\sin^2\theta + \lambda^{-2}\cos^2\theta &= 1 + \gamma^2, \\ (\lambda^2 - \lambda^{-2})\sin\theta\cos\theta &= \gamma, \end{aligned}$$

from which we may deduce that

$$\tan 2\theta = -\frac{2}{\gamma} \quad (\frac{\pi}{4} \le \theta < \frac{\pi}{2}).$$



The corresponding angle for the principal axes of $\mathbf{F}\mathbf{F}^T = \mathbf{V}^2$ is calculated in a similar way. Let $\mathbf{v}^{(1)} = \cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2$, $\mathbf{v}^{(2)} = -\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2$. The result is

$$\tan 2\phi = \frac{2}{\gamma} \quad (0 < \phi \le \frac{\pi}{4}).$$

3.8 Analysis of motion

The *velocity gradient tensor*, denoted $\mathbf{L}$, is defined as

$$\mathbf{L} = \operatorname{grad}\mathbf{v} \tag{3.40}$$

and has components

$$L_{ij} = \frac{\partial v_i}{\partial x_j} \tag{3.41}$$

with respect to the basis $\{\mathbf{e}_i\}$.

Using the second equation in (3.24) we obtain

$$\mathrm{Grad}\,\mathbf{v} = (\mathrm{grad}\,\mathbf{v})\mathbf{F} = \mathbf{LF}.$$

Since $\mathbf{v} = \dot{\mathbf{x}}$ we also have

$$\mathrm{Grad}\,\dot{\mathbf{x}} = \frac{\partial}{\partial t}\mathrm{Grad}\,\mathbf{x} = \dot{\mathbf{F}}.$$

Hence, we have the important connection

$$\dot{\mathbf{F}} = \mathbf{LF}. \tag{3.42}$$

Using (3.26)

$$\frac{\partial}{\partial t}(\det \mathbf{F}) = (\det \mathbf{F})\mathrm{tr}\,(\mathbf{F}^{-1}\dot{\mathbf{F}})$$

together with (3.42) we deduce that

$$\frac{\partial}{\partial t}(\det \mathbf{F}) = (\det \mathbf{F})\mathrm{tr}\,(\mathbf{L})$$

or, equivalently,

$$\dot{J} = J\mathrm{tr}\,(\mathbf{L}) = J\mathrm{div}\,\mathbf{v}, \tag{3.43}$$

remembering that $J = \det \mathbf{F}$, $\mathrm{tr}\,(\mathbf{L}) = L_{ii} = \partial v_i/\partial x_i = \mathrm{div}\,\mathbf{v}$.

Thus, $\mathrm{div}\,\mathbf{v}$ measures the rate at which volume changes during the motion.

For an *isochoric* motion $J \equiv 1$, $\dot{J} = 0$ and hence

$$\mathrm{div}\,\mathbf{v} = 0. \tag{3.44}$$

3.8.1 Stretching and spin

The deformation gradient $\mathbf{F}$ describes how material line elements change their length and orientation during deformation; the velocity gradient $\mathbf{L}$ describes the rate of these changes. Note that while $\mathbf{F}$ relates B_t to B_r, $\mathbf{L}$ is independent of B_r.

Write

$$\mathbf{L} = \mathbf{D} + \mathbf{W}, \tag{3.46}$$

where

$$\mathbf{D} = \underbrace{\frac{1}{2}(\mathbf{L} + \mathbf{L}^T)}_{\text{symmetric}}, \quad \mathbf{W} = \underbrace{\frac{1}{2}(\mathbf{L} - \mathbf{L}^T)}_{\text{skewsymmetric}}. \tag{3.47}$$

In order to interpret $\mathbf{D}$ and $\mathbf{W}$ we consider the line element $d\mathbf{X} \to d\mathbf{x} = \mathbf{F}d\mathbf{X}$. Form the difference

$$\begin{aligned} d\mathbf{x} \cdot d\mathbf{x} - d\mathbf{X} \cdot d\mathbf{X} &= (\mathbf{F}d\mathbf{X}) \cdot (\mathbf{F}d\mathbf{X}) - d\mathbf{X} \cdot d\mathbf{X} \\ &= d\mathbf{X} \cdot (\mathbf{F}^T\mathbf{F}d\mathbf{X}) - d\mathbf{X} \cdot d\mathbf{X} \\ &= d\mathbf{X} \cdot (\mathbf{F}^T\mathbf{F} - \mathbf{I})d\mathbf{X}. \end{aligned}$$

From (3.42) it follows that

$$\begin{aligned}\tfrac{\partial}{\partial t}(d\mathbf{x}\cdot d\mathbf{x}-d\mathbf{X}\cdot d\mathbf{X}) &= d\mathbf{X}\cdot\tfrac{\partial}{\partial t}(\mathbf{F}^T\mathbf{F}-\mathbf{I})d\mathbf{X} = d\mathbf{X}\cdot(\mathbf{F}^T\dot{\mathbf{F}}+\dot{\mathbf{F}}^T\mathbf{F})d\mathbf{X}\\ &= d\mathbf{X}\cdot(\mathbf{F}^T\mathbf{L}\mathbf{F}+\mathbf{F}^T\mathbf{L}^T\mathbf{F})d\mathbf{X} = (\mathbf{F}d\mathbf{X})\cdot(\mathbf{L}+\mathbf{L}^T)\mathbf{F}d\mathbf{X}\\ &= 2d\mathbf{x}\cdot(\mathbf{D}d\mathbf{x}).\end{aligned}$$

This shows that $\mathbf{D}$ measures the rate at which line elements are changing their lengths. It is called the (*Eulerian*) *strain-rate tensor* or *rate of stretching tensor*. The motion is *rigid* if and only if $\mathbf{D}=\mathbf{O}$.

Since

$$\frac{\partial}{\partial t}d\mathbf{x} = \dot{\mathbf{F}}d\mathbf{X} = \mathbf{L}\mathbf{F}d\mathbf{X} = \mathbf{L}d\mathbf{x} = (\mathbf{D}+\mathbf{W})d\mathbf{x}$$

and we have an interpretation of $\mathbf{D}$, as discussed above, it remains to interpret $\mathbf{W}$. We do this by setting $\mathbf{D}=\mathbf{O}$, so that

$$\frac{\partial}{\partial t}d\mathbf{x} = \mathbf{W}d\mathbf{x} = \mathbf{w}\times d\mathbf{x},$$

where $\mathbf{w}$ is the axial vector of $\mathbf{W}$. This shows that the motion is locally a rigid rotation and $\mathbf{W}$ is a measure of the *rate of rotation* (or *spin*) of line elements — $\mathbf{W}$ is called the *body spin*. The combination of $\mathbf{D}$ and $\mathbf{W}$ shows that the motion consists of stretching and rotation (analogous to the interpretation of $\mathbf{U}$ and $\mathbf{R}$).

3.9 Integration of tensors

We first summarize some results from vector calculus which will be needed. The *divergence theorem* is written

$$\int_R \operatorname{div}\mathbf{v}dV = \int_{\partial R}\mathbf{v}\cdot\mathbf{n}dA, \tag{3.48}$$

where R is a domain in $\mathbb{R}^3$ and ∂R is its boundary (a closed surface), and $\mathbf{v}$ is a vector field. An alternative form of the theorem is

$$\int_R \nabla\phi dV = \int_{\partial R}\phi\mathbf{n}dA, \tag{3.49}$$

where ϕ is a scalar field, or, in index notation,

$$\int_R \frac{\partial\phi}{\partial x_i}dV = \int_{\partial R}\phi n_i dA. \tag{3.50}$$

In particular, (3.50) applies to the *components* (which are scalar fields) $T_{pqr\ldots}$ of any CT. Thus,

$$\int_R \frac{\partial T_{pqr\ldots}}{\partial x_i}dV = \int_{\partial R}T_{pqr\ldots}n_i dA. \tag{3.51}$$

In tensor notation (3.51) is equivalent to

$$\int_R \nabla\otimes\mathbf{T}dV = \int_{\partial R}\mathbf{T}\otimes\mathbf{n}dA. \tag{3.52}$$

If, in particular, $\mathbf{T}$ is a CT(2) then contraction of (3.51), putting $i = p$, gives

$$\int_R \frac{\partial T_{pq}}{\partial x_p} dV = \int_{\partial R} T_{pq} n_p dA \tag{3.53}$$

or, in tensor notation,

$$\int_R \operatorname{div} \mathbf{T} dV = \int_{\partial R} \mathbf{T}^T \mathbf{n} dA. \tag{3.54}$$

This is an *important formula* and will occur frequently in the remaining sections of the notes.

For completeness, we also recall Stokes' theorem

$$\int_C \mathbf{u} \cdot d\mathbf{x} = \int_S (\nabla \times \mathbf{u}) \cdot \mathbf{n} dA, \tag{3.55}$$

or its equivalent

$$\int_C \phi d\mathbf{x} = \int_S d\mathbf{A} \times \nabla\phi, \tag{3.56}$$

where S is an open surface bounded by the contour C, $\mathbf{u}$ is a vector field and $d\mathbf{A} = \mathbf{n} dA$.

3.10 Transport formulae

Let C_t, S_t and R_t denote curves, surfaces and regions in B_t, the current configuration of the body. Then, the following identities hold:

$$\frac{d}{dt}\int_{C_t} \phi d\mathbf{x} = \int_{C_t} (\dot{\phi} d\mathbf{x} + \phi \mathbf{L} d\mathbf{x}), \tag{3.57}$$

$$\frac{d}{dt}\int_{S_t} \phi \mathbf{n} da = \int_{S_t} \{[\dot{\phi} + \phi \mathrm{tr}\,(\mathbf{L})]\mathbf{n} - \phi \mathbf{L}^T \mathbf{n}\} da, \tag{3.58}$$

$$\frac{d}{dt}\int_{R_t} \phi dv = \int_{R_t} [\dot{\phi} + \phi \mathrm{tr}\,(\mathbf{L})] dv, \tag{3.59}$$

$$\frac{d}{dt}\int_{C_t} \mathbf{u} \cdot d\mathbf{x} = \int_{C_t} (\dot{\mathbf{u}} + \mathbf{L}^T \mathbf{u}) \cdot d\mathbf{x}, \tag{3.60}$$

$$\frac{d}{dt}\int_{S_t} \mathbf{u} \cdot \mathbf{n} da = \int_{S_t} [\dot{\mathbf{u}} + \mathbf{u} \mathrm{tr}\,(\mathbf{L}) - \mathbf{L}\mathbf{u}] \cdot \mathbf{n} da, \tag{3.61}$$

$$\frac{d}{dt}\int_{R_t} \mathbf{u} dv = \int_{R_t} [\dot{\mathbf{u}} + \mathrm{tr}\,(\mathbf{L})\mathbf{u}] dv. \tag{3.62}$$

Proof Use the formulae $d\mathbf{x} = \mathbf{F} d\mathbf{X}, \mathbf{n} da = J\mathbf{F}^{-T}\mathbf{N} dA, dv = JdV$ to convert the integrals over C_t, S_t, R_t in B_t to integrals over C_r, S_r, R_r in B_r, together with expressions for

$\dot{\mathbf{F}}$ and $\dot{J}$. We illustrate the process by proving (3.58).

$$\begin{aligned}
\tfrac{d}{dt}\int_{S_t}\mathbf{u}\cdot\mathbf{n}da &= \tfrac{d}{dt}\int_{S_r}\mathbf{u}\cdot(J\mathbf{F}^{-T}\mathbf{N})dA \\
&\quad\text{(note the integral is now over } S_r\text{)} \\
&= \tfrac{d}{dt}\int_{S_r}(J\mathbf{F}^{-1}\mathbf{u})\cdot\mathbf{N}dA \\
&\quad\text{(using the definition of transpose)} \\
&= \int_{S_r}\underbrace{\frac{\partial}{\partial t}(J\mathbf{F}^{-1}\mathbf{u})}_{\text{at fixed }\mathbf{X}}\cdot\mathbf{N}dA \\
&= \int_{S_r}[J\mathbf{F}^{-1}\dot{\mathbf{u}}+\dot{J}\mathbf{F}^{-1}\mathbf{u}+J\partial(\mathbf{F}^{-1})/\partial t\,\mathbf{u}]\cdot\mathbf{N}dA.
\end{aligned}$$

From (3.43) we have $\dot{J}=J\mathrm{tr}\,(\mathbf{L})$, and from (3.45) we have $\partial(\mathbf{F}^{-1})/\partial t=-\mathbf{F}^{-1}\mathbf{L}$. Thus,

$$\begin{aligned}
\tfrac{d}{dt}\int_{S_t}\mathbf{u}\cdot\mathbf{n}da &= \int_{S_r}[J\mathbf{F}^{-1}\dot{\mathbf{u}}+J\mathrm{tr}\,(\mathbf{L})\mathbf{F}^{-1}\mathbf{u}-J\mathbf{F}^{-1}\mathbf{L}\mathbf{u}]\cdot\mathbf{N}dA \\
&= \int_{S_r}\{J\mathbf{F}^{-1}[\dot{\mathbf{u}}+\mathrm{tr}\,(\mathbf{L})\mathbf{u}-\mathbf{L}\mathbf{u}]\}\cdot\mathbf{N}dA \\
&= \int_{S_r}[\dot{\mathbf{u}}+\mathrm{tr}\,(\mathbf{L})\mathbf{u}-\mathbf{L}\mathbf{u}]\cdot(J\mathbf{F}^{-T}\mathbf{N})dA \\
&= \int_{S_t}[\dot{\mathbf{u}}+\mathrm{tr}\,(\mathbf{L})\mathbf{u}-\mathbf{L}\mathbf{u}]\cdot\mathbf{n}da \\
&\quad\text{(converting back to an integral over } S_t\text{).}
\end{aligned}$$

We can establish the other formulae by following the same approach.

4 Balance laws and equations of motion

The mechanics of continuous media are described by equations which express the balance of mass, linear momentum, angular momentum and energy in a moving body. These balance equations apply to all bodies, solid or fluid, and each gives rise to field equations (differential equations for scalar, vector and tensor fields) for sufficiently smooth motions (or jump conditions across surfaces of discontinuity). The fundamental concepts are mass, force and energy.

4.1 Mass

Let B be an arbitrary configuration of a body $\mathcal{B}$, and let A be a set of points in B occupied by the particles in an arbitrary subset $\mathcal{A}$ of $\mathcal{B}$. If, with $\mathcal{A}$, there is associated a non-negative real number $m(\mathcal{A})$ having physical dimensions independent of time and distance, such that
(i) $m(\mathcal{A}_1 \cup \mathcal{A}_2) = m(\mathcal{A}_1) + m(\mathcal{A}_2)$

for all pairs $\mathcal{A}_1, \mathcal{A}_2$ of disjoint subsets of $\mathcal{B}$, and
(ii) $m(\mathcal{A}) \to 0$ as the volume of A tends to zero,
then $\mathcal{B}$ is said to be a body *with mass function* m. The mass of A is denoted $m(\mathrm{A})$.

Properties (i) and (ii) imply that there exists a scalar field ρ defined on B such that

$$m(\mathrm{A}) = \int_A \rho dv \tag{4.1}$$

(this is a result from measure theory, so a proof will not be given here).

We refer to ρ as the *mass density* of the material composing $\mathcal{B}$ — it is a scalar field, usually assumed to be continuously differentiable (although sometimes allowed to be discontinuous, but not in this course).

4.2 Mass conservation

Let R_t be an arbitrary material region in the current configuration B_t. As R_t moves it always consists of the same material, so its mass does not change, i.e.

$$\frac{d}{dt}\int_{R_t} \rho dv = 0 \tag{4.2}$$

— this is one form of the *conservation of mass equation.* From the transport formula (2.59) we obtain

$$\int_{R_t} (\dot{\rho} + \rho \mathrm{div}\, \mathbf{v}) dv = 0,$$

and, since R_t is arbitrary, it follows that

$$\dot{\rho} + \rho \mathrm{div}\, \mathbf{v} = 0 \tag{4.3}$$

at each point of the body (this deduction requires that the integrand is continuous). This is the local form of the mass conservation equation — it is also known as the *continuity equation*.

Recall, from (3.43), that

$$\dot{J} = J\text{div}\,\mathbf{v}.$$

Substitution for $\text{div}\,\mathbf{v}$ from (4.3) then gives

$$\rho\dot{J} + \dot{\rho}J = 0,$$

i.e. $\partial(\rho J)/\partial t = 0$. Thus, ρJ is constant for any material particle. In the reference configuration $J = 1$, so that $\rho J = \rho_r$, where ρ_r is the mass density in the reference configuration. Thus,

$$\rho = J^{-1}\rho_r \tag{4.4}$$

— another form of the mass conservation equation.

An alternative way to derive (4.4) is to note that

$$\int_{R_t} \rho dv = \int_{R_r} \rho J dV = \int_{R_r} \rho_r dV.$$

4.3 Force, momentum and torque

Let $R_t \subset \mathrm{B}_t$. The *linear momentum* of the material occupying R_t is defined as

$$\mathbf{M}(R_t) = \int_{R_t} \rho\mathbf{v}dv. \tag{4.5}$$

If $\mathbf{x}$ is a point of R_t with respect to an origin o then the *angular momentum* of R_t with respect to o is

$$\mathbf{H}(R_t; o) = \int_{R_t} \mathbf{x} \times (\rho\mathbf{v})dv. \tag{4.6}$$

4.3.1 Application to rigid motions

Recall, from (2.4), that a rigid motion is given by

$$\mathbf{x} = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{X}.$$

It follows that

$$\mathbf{v} \equiv \dot{\mathbf{x}} = \dot{\mathbf{c}} + \dot{\mathbf{Q}}\mathbf{X} = \dot{\mathbf{c}} + \dot{\mathbf{Q}}\mathbf{Q}^T(\mathbf{x} - \mathbf{c}) = \dot{\mathbf{c}} + \mathbf{W}(\mathbf{x} - \mathbf{c}),$$

where $\mathbf{W} = \dot{\mathbf{Q}}\mathbf{Q}^T$ is skewsymmetric. Hence

$$\mathbf{v} = \dot{\mathbf{c}} + \mathbf{w} \times (\mathbf{x} - \mathbf{c}),$$

where $\mathbf{w}$ is the axial vector of $\mathbf{W}$. From (4.5) we therefore obtain

$$\mathbf{M} = \int_{R_t} \rho[\dot{\mathbf{c}} + \mathbf{w} \times (\mathbf{x} - \mathbf{c})]dv = m[\dot{\mathbf{c}} + \mathbf{w} \times (\bar{\mathbf{x}} - \mathbf{c})], \tag{4.7}$$

where

$$m = \int_{R_t} \rho dv,$$

the total mass in R_t, and

$$m\bar{\mathbf{x}} = \int_{R_t} \rho \mathbf{x} dv,$$

$\bar{\mathbf{x}}$ being the *centre of mass* of R_t relative to o.

Similarly,

$$\mathbf{H} = \int_{R_t} \rho \mathbf{x} \times [\dot{\mathbf{c}} + \mathbf{w} \times (\mathbf{x} - \mathbf{c})]dv = m\bar{\mathbf{x}} \times (\dot{\mathbf{c}} - \mathbf{w} \times \mathbf{c}) + \mathbf{K}\mathbf{w}, \tag{4.8}$$

where

$$\mathbf{K} = \int_{R_t} \rho[(\mathbf{x} \cdot \mathbf{x})\mathbf{I} - \mathbf{x} \otimes \mathbf{x}]dv \tag{4.9}$$

is the *inertia tensor* of the material in R_t relative to o.

Note that (4.7) and (4.8) may be written as

$$\mathbf{M} = m\bar{\mathbf{v}}, \qquad \mathbf{H} = m\bar{\mathbf{x}} \times \bar{\mathbf{v}} + \bar{\mathbf{K}}\mathbf{w}, \tag{4.10}$$

where $\bar{\mathbf{v}} \equiv \dot{\mathbf{c}} + \mathbf{w} \times (\bar{\mathbf{x}} - \mathbf{c})$ is the velocity of the centre of mass and $\bar{\mathbf{K}}$ is the inertia tensor relative to the centre of mass.

4.4 Body and surface forces

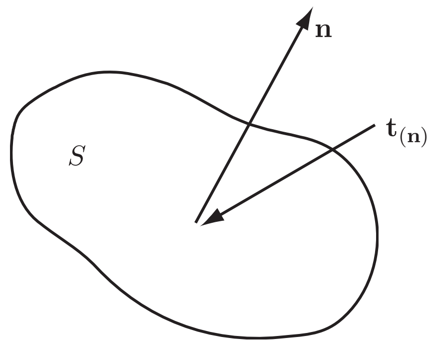
The concepts of *force* and *torque* describe the action on a moving material body $\mathcal{B}$ of its surroundings and the mutual actions of the parts of $\mathcal{B}$ on each other. With $R_t \subset \mathrm{B}_t$ we associate two vectors, $\mathbf{F}(R_t)$ and $\mathbf{G}(R_t; o)$, called the *force* and the *torque with respect to* o on the material in R_t. Two types of force and torque must be accounted for in general — *body forces* and *body torques*, which act on the particles of a body (arising from gravity or magnetic fields, for example), and *contact forces* and *contact torques* resulting from the action of one part of the body on another across a separating surface.

The *body* force and torque, measured *per unit mass*, are denoted $\mathbf{b}$ and $\mathbf{c}$ respectively. Their contributions to $\mathbf{F}(R_t)$ and $\mathbf{G}(R_t; o)$ are

$$\int_{R_t} \rho \mathbf{b} dv, \qquad \int_{R_t} [\mathbf{x} \times (\rho \mathbf{b}) + \rho \mathbf{c}]dv$$

respectively.

A mathematical description of contact forces (but not torques) relies on *Cauchy's stress principle* (which is regarded as an axiom). This states that
the action of the material occupying that part of B_t *exterior to a closed surface* S *on the material occupying the interior part is represented by a vector field* $\mathbf{t}_{(\mathbf{n})}$*, with physical dimensions of force per unit area, defined on* S*.*



We refer to $\mathbf{t}_{(\mathbf{n})}$ as the *stress vector*. It is assumed to depend continuously on $\mathbf{n}$, the unit outward normal to S.

If this stress principle gives a complete account of contact action then the material is said to be *non-polar* and does not admit contact torques. All classical theories of solids and fluids are of this type.

The contributions to $\mathbf{F}(R_t)$ and $\mathbf{G}(R_t; o)$ of the contact forces acting on the boundary ∂R_t of R_t are

$$\int_{\partial R_t} \mathbf{t}_{(\mathbf{n})} da, \qquad \int_{\partial R_t} \mathbf{x} \times \mathbf{t}_{(\mathbf{n})} da$$

respectively. We now have

$$\begin{aligned} \mathbf{F}(R_t) &= \textstyle\int_{R_t} \rho \mathbf{b} dv + \int_{\partial R_t} \mathbf{t}_{(\mathbf{n})} da, \\ \mathbf{G}(R_t; o) &= \textstyle\int_{R_t} \rho(\mathbf{x} \times \mathbf{b} + \mathbf{c}) dv + \int_{\partial R_t} \mathbf{x} \times \mathbf{t}_{(\mathbf{n})} da \end{aligned} \tag{4.11}$$

for the total force and torque (sometimes referred to as couple) about o acting on R_t.

4.5 Euler's laws of motion

Euler's laws of motion are

$$\frac{d\mathbf{M}}{dt} = \mathbf{F}, \qquad \frac{d\mathbf{H}}{dt} = \mathbf{G}, \tag{4.12}$$

and these hold independently of the choice of origin (although $\mathbf{G}$ and $\mathbf{H}$ do depend on such a choice).

They parallel Newton's laws for particles and rigid bodies. Note, however, that in classical mechanics $(4.12)_2$ is a consequence of $(4.12)_1$, whereas in continuum mechanics this is not the case – the two equation (4.12) are *independent*.

Normally, body torques do not arise, so we set $\mathbf{c} = \mathbf{0}$, and (4.12) are then written in full as

$$\begin{aligned} \frac{d}{dt}\int_{R_t} \rho\mathbf{v}dv &= \textstyle\int_{R_t} \rho\mathbf{b}dv + \int_{\partial R_t} \mathbf{t}_{(\mathbf{n})}da, \\ \frac{d}{dt}\int_{R_t} \rho\mathbf{x}\times\mathbf{v}dv &= \textstyle\int_{R_t} \rho\mathbf{x}\times\mathbf{b}dv + \int_{\partial R_t} \mathbf{x}\times\mathbf{t}_{(\mathbf{n})}da, \end{aligned} \tag{4.13}$$

— the equations of *linear* and *angular momentum balance*.

Using the transport formula (2.62) with $\mathbf{u} = \rho\mathbf{v}$ and (4.3) we obtain

$$\frac{d}{dt}\int_{R_t} \rho\mathbf{v}dv = \int_{R_t} [\rho\dot{\mathbf{v}} + \dot{\rho}\mathbf{v} + \rho(\operatorname{div}\mathbf{v})\mathbf{v}]dv = \int_{R_t} \rho\dot{\mathbf{v}}dv.$$

Hence (4.13) can be written

$$\begin{aligned} \int_{R_t} \rho(\mathbf{a}-\mathbf{b})dv &= \textstyle\int_{\partial R_t} \mathbf{t}_{(\mathbf{n})}da, \\ \int_{R_t} \rho\mathbf{x}\times(\mathbf{a}-\mathbf{b})dv &= \textstyle\int_{\partial R_t} \mathbf{x}\times\mathbf{t}_{(\mathbf{n})}da, \end{aligned} \tag{4.14}$$

where $\mathbf{a} \equiv \dot{\mathbf{v}}$ is the acceleration.

4.6 The theory of stress

4.6.1 Cauchy's theorem

Let $(\mathbf{t}, \mathbf{b})$ be a system of surface (contact) and body forces for $\mathcal{B}$ during a motion. A necessary and sufficient condition for the momentum balance equations (4.14) to be satisfied is that there exists a second-order tensor $\boldsymbol{\sigma}$ (called the *Cauchy stress tensor*) such that (i) for each unit vector $\mathbf{n}$,

$$\mathbf{t}_{(\mathbf{n})} = \boldsymbol{\sigma}^T\mathbf{n}, \tag{4.15}$$

where $\boldsymbol{\sigma}$ is independent of $\mathbf{n}$, (ii)

$$\boldsymbol{\sigma}^T = \boldsymbol{\sigma}, \tag{4.16}$$

(iii) $\boldsymbol{\sigma}$ satisfies the *equation of motion*

$$\operatorname{div}\boldsymbol{\sigma} + \rho\mathbf{b} = \rho\mathbf{a}. \tag{4.17}$$

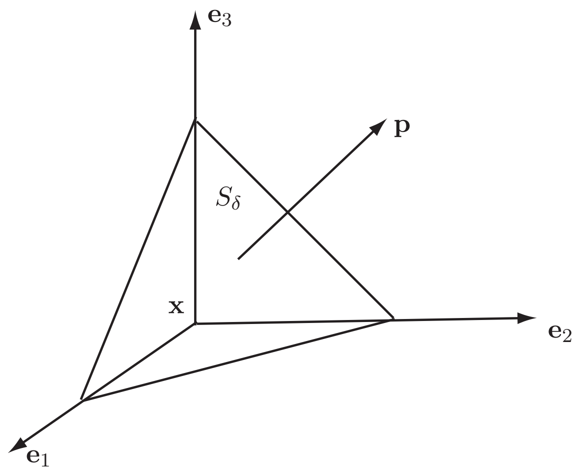
Proof Sufficiency: check by substituting (4.15)–(4.17) into (4.14). The calculations involved are similar to those required to prove necessity, so will not be given here. Necessity: assume that (4.14) are satisfied. The proof involves a number of steps.

Lemma 1

Given any $\mathbf{x} \in \mathrm{B}_t$, any orthonormal basis $\{\mathbf{e}_i\}$ and any vector $\mathbf{p}$ with $\mathbf{p}\cdot\mathbf{e}_i > 0, i \in \{1,2,3\}$, we have

$$\mathbf{t}(\mathbf{p},\mathbf{x}) = -\sum_{i=1}^{3}(\mathbf{p}\cdot\mathbf{e}_i)\mathbf{t}(-\mathbf{e}_i,\mathbf{x}).$$

Proof



Let $\mathbf{x} \in \mathrm{B}_t$, $\delta > 0$ and consider the tetrahedron shown. The faces are S_1, S_2, S_3 and S_δ, with unit (outward) normals $-\mathbf{e}_1$, $-\mathbf{e}_2$, $-\mathbf{e}_3$, $\mathbf{p}$ respectively, δ being the distance of the sloping plane from $\mathbf{x}$.

Since $\mathbf{a}$ and $\mathbf{b}$ are continuous on B_t they are bounded on some neighbourhood of $\mathbf{x}$ in B_t containing the tetrahedron for sufficiently small δ. Similarly, ρ is bounded, so

$$\left|\int_{\partial R_t}\mathbf{t}_{(\mathbf{n})}da\right| = \left|\int_{R_t}\rho(\mathbf{a}-\mathbf{b})dv\right| < k\,\mathrm{vol}\,(R_t),$$

where k is a constant independent of δ.

If $A(\delta)$ denotes the area of the face S_δ then there exist positive constants c_1, c_2 such that

$$A(\delta) = c_1\delta^2, \quad \mathrm{vol}\,(R_t) = c_2\delta^3.$$

Hence

$$\frac{1}{A(\delta)}\int_{\partial R_t}\mathbf{t}_{(\mathbf{n})}da \to 0 \quad \text{as } \delta \to 0.$$

Let A_i denote the area of S_i. Since, by the divergence theorem, we have

$$\int_{\partial R_t}\mathbf{e}_i \cdot \mathbf{n} dS = 0 \quad i \in \{1,2,3\},$$

it follows that

$$A_i = (\mathbf{e}_i \cdot \mathbf{p})A(\delta).$$

But

$$\int_{\partial R_t}\mathbf{t}_{(\mathbf{n})}da = \int_{S_\delta}\mathbf{t}(\mathbf{p})da + \sum_{i=1}^{3}\int_{S_i}\mathbf{t}(-\mathbf{e}_i)da$$

and

$$\frac{1}{A(\delta)}\int_{S_\delta}\mathbf{t}(\mathbf{p})da \to \mathbf{t}(\mathbf{p},\mathbf{x}) \quad \text{as } \delta \to 0,$$

$$\frac{1}{A(\delta)}\int_{S_i}\mathbf{t}(-\mathbf{e}_i)da \to (\mathbf{e}_i \cdot \mathbf{p})\mathbf{t}(-\mathbf{e}_i,\mathbf{x}) \quad \text{as } \delta \to 0.$$

Hence the stated result.

It follows that

$$\mathbf{t}(\mathbf{e}_i,\mathbf{x}) = -\mathbf{t}(-\mathbf{e}_i,\mathbf{x})$$

and hence

$$\mathbf{t}(\mathbf{p},\mathbf{x}) = \sum_{i=1}^{3}(\mathbf{e}_i \cdot \mathbf{p})\mathbf{t}(\mathbf{e}_i,\mathbf{x}) \tag{4.18}$$

for *any* vector $\mathbf{p}$.

Main result consider the tensor $\boldsymbol{\sigma}$ defined by

$$\boldsymbol{\sigma}^T(\mathbf{x}) = \sum_{i=1}^{3}\mathbf{t}(\mathbf{e}_i,\mathbf{x}) \otimes \mathbf{e}_i. \tag{4.19}$$

Then

$$\boldsymbol{\sigma}^T\mathbf{n} = \sum_{i=1}^{3}(\mathbf{e}_i \cdot \mathbf{n})\mathbf{t}(\mathbf{e}_i,\mathbf{x}) = \mathbf{t}(\mathbf{n},\mathbf{x})$$

by (4.18). Hence (i) is established.

On substitution of $\mathbf{t}_{(\mathbf{n})} = \boldsymbol{\sigma}^T\mathbf{n}$ into $(4.14)_1$, we obtain

$$\int_{R_t}\rho(\mathbf{a}-\mathbf{b})dv = \int_{\partial R_t}\boldsymbol{\sigma}^T\mathbf{n}da = \int_{R_t}\operatorname{div}\boldsymbol{\sigma}dv$$

by the divergence theorem (2.54). Thus,

$$\int_{R_t} [\operatorname{div} \boldsymbol{\sigma} - \rho(\mathbf{a} - \mathbf{b})] dv = \mathbf{0}.$$

Since R_t is arbitrary, (iii) follows (provided the above integrand is continuous). It remains to prove (ii).

Next, substitute (4.15) and (4.17) into $(4.14)_2$ to give

$$\int_{R_t} \mathbf{x} \times (\operatorname{div} \boldsymbol{\sigma}) dv = \int_{\partial R_t} \mathbf{x} \times (\boldsymbol{\sigma}^T \mathbf{n}) da. \qquad (*)$$

Noting that $\mathbf{u} \times \mathbf{v} = \mathbf{a} \times \mathbf{b}$, for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{a}, \mathbf{b}$, is equivalent to

$$\mathbf{u} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{u} = \mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a},$$

we write $(*)$ as

$$\int_{R_t} [\mathbf{x} \otimes \operatorname{div} \boldsymbol{\sigma} - (\operatorname{div} \boldsymbol{\sigma}) \otimes \mathbf{x}] dv = \int_{\partial R_t} (\mathbf{x} \otimes \boldsymbol{\sigma}^T \mathbf{n} - \boldsymbol{\sigma}^T \mathbf{n} \otimes \mathbf{x}) da,$$

which, by application of the divergence theorem, becomes

$$\int_{R_t} [\mathbf{x} \otimes \operatorname{div} \boldsymbol{\sigma} + (\operatorname{grad} \mathbf{x}) \boldsymbol{\sigma} - (\operatorname{div} \boldsymbol{\sigma}) \otimes \mathbf{x} - \boldsymbol{\sigma}^T (\operatorname{grad} \mathbf{x})^T] dv.$$

Since $\operatorname{grad} \mathbf{x} = \mathbf{I}$, we deduce that

$$\int_{R_t} (\boldsymbol{\sigma} - \boldsymbol{\sigma}^T) dv = \mathbf{O}.$$

Since R_t is arbitrary, (ii) follows.

(It may prove easier to work through the above calculation using index notation.)

4.6.2 Normal and shear stresses

Suppose an element of area da on a surface S is subjected to a contact force $\mathbf{t}(\mathbf{n})da$. The *normal component* σ of the stress vector, defined as

$$\sigma \equiv \mathbf{n} \cdot \mathbf{t} = \mathbf{n} \cdot (\boldsymbol{\sigma} \mathbf{n}), \qquad (4.20)$$

is called the *normal stress* — it is tensile (compressive) when positive (negative).

The component of the stress vector tangential to S is

$$\tau \equiv |\mathbf{t} - (\mathbf{t} \cdot \mathbf{n})\mathbf{n}|, \qquad (4.21)$$

so that

$$\tau^2 = \mathbf{t} \cdot \mathbf{t} - (\mathbf{t} \cdot \mathbf{n})^2.$$

We refer to τ as the *shear stress*.

If $\tau = 0$ and σ is independent of $\mathbf{n}$ then the stress is said to be *hydrostatic*. In this case there is a scalar field p, called the *pressure*, such that

$$\mathbf{t}(\mathbf{n}) = -p\mathbf{n}, \qquad \boldsymbol{\sigma} = -p\mathbf{I}. \tag{4.22}$$

At a point $\mathbf{x}$ in the current configuration B_t let $\boldsymbol{\sigma}$ have components σ_{ij} with respect to basis vectors $\{\mathbf{e}_i\}$. Then σ_{ij} is the jth component of force per unit area in B_t acting on a surface whose normal is in the i-direction.

In the case of (4.22) $\sigma_{21} = 0$, $\sigma_{23} = 0$, $\sigma_{22} = -p$, so the only force acting on the shaded surface is a normal pressure.

Example
Let B_t be defined by

$$-a \leq x_1 \leq a, \quad -a \leq x_2 \leq a, \quad -h \leq x_3 \leq h.$$

Given that the stress field has components

$$\sigma_{11} = -\sigma_{22} = -q(x_1^2 - x_2^2)/a^2, \quad \sigma_{12} = 2qx_1x_2/a^2, \quad \sigma_{3i} = 0,$$

show that the equilibrium equations are satisfied and determine the forces which must be applied to $\partial\mathrm{B}_t$ to maintain this state of stress. Also, calculate the resultant forces and torques (about o) on the faces $x_1 = \pm a, x_2 = \pm a$.

We have

$$\begin{aligned} \frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} &= -2qx_1/a^2 + 2qx_1/a^2 = 0, \\ \frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} &= 2qx_2/a^2 - 2qx_2/a^2 = 0. \end{aligned}$$

Hence, $\operatorname{div}\boldsymbol{\sigma} = \mathbf{0}$, i.e. the equilibrium equation is satisfied.

On the faces $x_1 = \pm a$ we have $\mathbf{n} = \pm\mathbf{e}_1$, and hence

$$\begin{aligned} \mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \pm\boldsymbol{\sigma}\mathbf{e}_1 = \pm\sigma_{i1}\mathbf{e}_i &= \pm(\sigma_{11}, \sigma_{21}, \sigma_{31}) \\ &= \pm\big(-q(a^2 - x_2^2)/a^2, \pm 2qx_2/a, 0\big). \end{aligned}$$

Similarly, on $x_2 = \pm a$,

$$\mathbf{t} = \pm(\sigma_{12}, \sigma_{22}, \sigma_{32}) = \pm\big(\pm 2qx_1/a, q(x_1^2 - a^2)/a^2, 0\big).$$

On $x_3 = \pm h$ we have $\mathbf{t} = \mathbf{0}$.

The resultant force of $x_1 = a$ is

$$\int_S \mathbf{t} da = \textstyle\int_{-a}^{a} \big(-q(a^2 - x_2^2)/a^2, 2qx_2/a, 0\big) dx_2 \int_{-h}^{h} dx_3$$
$$= (-8qha/3, 0, 0).$$

The resultant torque on $x_1 = a$ is

$$\int_S \mathbf{x} \times \mathbf{t} da = \int_{-h}^{h} \int_{-a}^{a} \big(-2qx_2x_3/a, -q(a^2 - x_2^2)x_3/a^2, q(3a^2 - x_2^2)x_2/a^2\big) dx_2 dx_3 = \mathbf{0}.$$

Similarly for the other faces.

4.7 Energy

The *kinetic energy* $K(R_t)$ of the material occupying R_t is defined as

$$K(R_t) = \int_{R_t} \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} dv, \tag{4.23}$$

and the *rate of working*, or *power*, $P(R_t)$ of the forces acting on R_t is defined as

$$P(R_t) = \int_{R_t} \rho \mathbf{b} \cdot \mathbf{v} dv + \int_{\partial R_t} \mathbf{t} \cdot \mathbf{v} da. \tag{4.24}$$

By using $\mathbf{t} = \boldsymbol{\sigma} \mathbf{n}$ in (4.24), we obtain

$$\begin{aligned}
P(R_t) &= \textstyle\int_{R_t} \rho \mathbf{b} \cdot \mathbf{v} dv + \int_{\partial R_t} (\boldsymbol{\sigma} \mathbf{n}) \cdot \mathbf{v} da \\
&= \textstyle\int_{R_t} \rho \mathbf{b} \cdot \mathbf{v} dv + \int_{\partial R_t} (\boldsymbol{\sigma} \mathbf{v}) \cdot \mathbf{n} da \\
&\qquad \text{(since } \boldsymbol{\sigma} \text{ is symmetric)} \\
&= \textstyle\int_{R_t} [\rho \mathbf{b} \cdot \mathbf{v} + \operatorname{div}(\boldsymbol{\sigma} \mathbf{v})] dv \\
&\qquad \text{(by the divergence theorem)} \\
&= \textstyle\int_{R_t} [\rho \mathbf{b} \cdot \mathbf{v} + (\operatorname{div} \boldsymbol{\sigma}) \cdot \mathbf{v} + \operatorname{tr}(\boldsymbol{\sigma} \mathbf{L})] dv \\
&\qquad \text{(since } (\sigma_{ij} v_j)_{,i} = \sigma_{ij,i} v_j + \sigma_{ij} v_{j,i} = \sigma_{ij,i} v_j + \sigma_{ij} L_{ji}) \\
&= \textstyle\int_{R_t} [(\rho \mathbf{b} + \operatorname{div} \boldsymbol{\sigma}) \cdot \mathbf{v} + \operatorname{tr}(\boldsymbol{\sigma} \mathbf{D})] dv \\
&\qquad \text{(since } \sigma_{ij} L_{ji} = \sigma_{ij}(D_{ij} + W_{ij}) = \sigma_{ij} D_{ij}) \\
&= \textstyle\int_{R_t} [\rho \dot{\mathbf{v}} \cdot \mathbf{v} + \operatorname{tr}(\boldsymbol{\sigma} \mathbf{D})] dv \\
&\qquad \text{(using the equation of motion)} \\
&= \textstyle\int_{R_t} \frac{1}{2} \rho \partial(\mathbf{v} \cdot \mathbf{v})/\partial t \, dv + \int_{R_t} \operatorname{tr}(\boldsymbol{\sigma} \mathbf{D}) dv \\
&= \textstyle\int_{R_r} \frac{1}{2} \rho_r \partial(\mathbf{v} \cdot \mathbf{v})/\partial t \, dV + \int_{R_t} \operatorname{tr}(\boldsymbol{\sigma} \mathbf{D}) dv \\
&\qquad \text{(since } \rho dv = \rho_r dV) \\
&= \textstyle\frac{d}{dt} \int_{R_r} \frac{1}{2} \rho_r (\mathbf{v} \cdot \mathbf{v}) dV + \int_{R_t} \operatorname{tr}(\boldsymbol{\sigma} \mathbf{D}) dv \\
&= \textstyle\frac{d}{dt} K(R_t) + \int_{R_t} \operatorname{tr}(\boldsymbol{\sigma} \mathbf{D}) dv.
\end{aligned}$$

Thus,

$$P(R_t) = \frac{d}{dt}K(R_t) + S(R_t), \tag{4.25}$$

where

$$S(R_t) = \int_{R_t} \mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{D})dv. \tag{4.26}$$

Equation (4.25) is an energy balance equation – work done by the body and surface forces is converted into kinetic energy and $S(R_t)$ — the latter may consist of stored (or potential) energy or be a measure of the amount of work dissipated in the form of heat or be a mixture of the two.

5 Constitutive equations

5.1 Constitutive assumptions

So far, we have the following equations governing the motion of a continuous body:

equation of mass conservation

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0; \tag{5.1}$$

equation of motion

$$\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}; \tag{5.2}$$

equation of angular momentum balance

$$\boldsymbol{\sigma}^T = \boldsymbol{\sigma}. \tag{5.3}$$

These provide 7 scalar equations for 13 scalar fields — $\rho, \mathbf{v}$ (3 components), $\boldsymbol{\sigma}$ (9 components) with the body force $\mathbf{b}$ regarded as known. Equivalently, given (5.3), equations (5.1) and (5.2) provide 4 equations for 10 scalar fields — $\rho, \mathbf{v}$ (3 components), $\boldsymbol{\sigma}$ (6 components).

The missing 6 equations are provided in the form of *constitutive equations*, which give expressions for the 6 components of $\boldsymbol{\sigma}$ in terms of kinematical quantities, as we now describe.

It is assumed that at time t the stress is uniquely determined by the motion $\boldsymbol{\chi}$, i.e. $\boldsymbol{\sigma}$ is a function (more generally functional) of $\mathbf{x}, \mathbf{v}, \mathbf{F}, \mathbf{L}, \ldots$, and possibly also higher gradients of the deformation. We then have 10 equations for 10 unknowns, and, by substituting the constitutive equations into (5.1) and (5.2) we arrive at 4 equations for ρ and $\mathbf{v}$, and (5.3) will be satisfied automatically. We now illustrate the general principles involved in the development of constitutive equations by focussing on the case of homogeneous elastic materials, for which $\boldsymbol{\sigma}$ depends only on $\mathbf{F}$. (For an inhomogeneous material there is, additionally, explicit dependence on $\mathbf{X}$.)

5.2 Elastic materials

The constitutive equation for an elastic material is written in the form

$$\boldsymbol{\sigma} = \mathbf{g}(\mathbf{F}), \tag{5.4}$$

where $\mathbf{g}$ is a *symmetric tensor-valued function*, defined on the space of deformation gradients $\mathbf{F}$. Equation (5.4) states that the stress in B_t at a point $\mathbf{X}$ depends only on the deformation gradient at $\mathbf{X}$ and not on the history of deformation, and, in particular, it is independent of the path of deformation taken to reach the point $\mathbf{F}$.

When the stress is removed the deformation returns to its original value (that in B_r), so that

$$\mathbf{g}(\mathbf{I}) = \mathbf{O}, \tag{5.5}$$

i.e. the undeformed configuration is free of stress. In some situations it will be necessary to relax this condition, but it is adopted here for the time being. We refer to $\mathbf{g}$ as the (Cauchy stress) *response function* of the material *relative* to B_r. It should be emphasized that, in general, the form of $\mathbf{g}$ depends on the choice of reference configuration.

5.3 Objectivity

Suppose that a rigid-body motion

$$\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x} + \mathbf{c}(t) \tag{5.6}$$

is superimposed on the motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$. Then, the resulting deformation gradient, $\mathbf{F}^*$ say, is given by

$$\mathbf{F}^* = \mathbf{QF}. \tag{5.7}$$

In index notation, this may be proved as follows. Note first that, since

$$x_i^* = Q_{ip}x_p + c_i,$$

we obtain

$$\frac{\partial x_i^*}{\partial x_k} = Q_{ip}\frac{\partial x_p}{\partial x_k} = Q_{ip}\delta_{pk} = Q_{ik},$$

and hence

$$F_{ij}^* = \frac{\partial x_i^*}{\partial X_j} = \frac{\partial x_i^*}{\partial x_k}\frac{\partial x_k}{\partial X_j} = Q_{ik}F_{kj}.$$

For an elastic material with response function $\mathbf{g}$, the stress tensor, $\boldsymbol{\sigma}^*$ say, associated with the deformation gradient $\mathbf{F}^*$ is

$$\boldsymbol{\sigma}^* = \mathbf{g}(\mathbf{F}^*).$$

We now determine how $\boldsymbol{\sigma}^*$ is related to $\boldsymbol{\sigma}$. Under the rotation $\mathbf{Q}$ the unit normal $\mathbf{n}$ to ∂R_t becomes $\mathbf{n}^* = \mathbf{Qn}$ and the traction vector $\mathbf{t}$ becomes $\mathbf{t}^* = \mathbf{Qt}$. Since $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$, $\mathbf{t}^* = \boldsymbol{\sigma}^*\mathbf{n}^*$ we obtain

$$\mathbf{Q}\boldsymbol{\sigma}\mathbf{n} = \boldsymbol{\sigma}^*\mathbf{Qn}.$$

This holds for arbitrary $\mathbf{n}$ and hence $\mathbf{Q}\boldsymbol{\sigma} = \boldsymbol{\sigma}^*\mathbf{Q}$, i.e.

$$\boldsymbol{\sigma}^* = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T. \tag{5.8}$$

The response function $\mathbf{g}$ must therefore satisfy the *invariance requirement*

$$\mathbf{g}(\mathbf{F}^*) \equiv \mathbf{g}(\mathbf{QF}) = \mathbf{Qg}(\mathbf{F})\mathbf{Q}^T \tag{5.9}$$

for each $\mathbf{F}$ and *all* rotations $\mathbf{Q}$. This expresses the fact that the constitutive law (5.4) is *objective*. In essence, this means that material properties are independent of superimposed rigid-body motions. It can also be interpreted in terms of *observers*. In that case, rather than representing a superimposed rigid motion equation, (5.6) is treated as an *observer transformation*. For the elastic materials discussed here the consequences of the two interpretations are identical, but in general this may not be the case for other materials. The difference is quite subtle and has generated some controversy in the literature.

Definition 1. *Let $\phi, \mathbf{u}, \mathbf{T}$ be scalar, vector and (second-order) tensor fields defined on B_t, i.e. they are Eulerian in character. Let $\phi^*, \mathbf{u}^*, \mathbf{T}^*$ be the corresponding fields defined on B_t^*, where B_t^* is obtained from B_t by the rigid motion $\mathbf{x}^* = \mathbf{Q}\mathbf{x} + \mathbf{c}$. The fields are said to be objective if, for all such motions,*

$$\phi^* = \phi, \quad \mathbf{u}^* = \mathbf{Q}\mathbf{u}, \quad \mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T. \tag{5.10}$$

Example If ϕ is an objective scalar field then $\operatorname{grad}\phi$ is an objective vector field. We note that, in components,

$$\begin{aligned}&(\operatorname{grad}\phi)_i^* = (\operatorname{grad}^*\phi^*)_i = (\operatorname{grad}^*\phi)_i \qquad (\text{since } \phi^* = \phi)\\ &= \frac{\partial \phi}{\partial x_i^*} = \frac{\partial \phi}{\partial x_k}\frac{\partial x_k}{\partial x_i^*}.\end{aligned}$$

Next, since $\mathbf{x}^* = \mathbf{Q}\mathbf{x} + \mathbf{c}$, it follows that $\mathbf{x} = \mathbf{Q}^T\mathbf{x}^* - \mathbf{Q}^T\mathbf{c}$, or, in components,

$$x_k = Q_{pk}x_p^* - Q_{pk}c_p.$$

Hence

$$\frac{\partial x_k}{\partial x_i^*} = Q_{pk}\frac{\partial x_p^*}{\partial x_i^*} = Q_{pk}\delta_{pi} = Q_{ik},$$

which leads to

$$(\operatorname{grad}\phi)_i^* = Q_{ik}(\operatorname{grad}\phi)_k,$$

i.e. $(\operatorname{grad}\phi)^* = \mathbf{Q}(\operatorname{grad}\phi)$. Thus, $\operatorname{grad}\phi$ is an objective vector.

Example Let

$$\mathbf{x}^* \equiv \boldsymbol{\chi}^*(\mathbf{X}, t) = \mathbf{Q}(t)\boldsymbol{\chi}(\mathbf{X}, t) + \mathbf{c}(t).$$

Then

$$\mathbf{v}^* \equiv \frac{\partial \boldsymbol{\chi}^*}{\partial t} = \mathbf{Q}\mathbf{v} + \dot{\mathbf{Q}}\mathbf{x} + \dot{\mathbf{c}}.$$

Since $\mathbf{v}^* \neq \mathbf{Q}\mathbf{v}$ it follows that the velocity is *not* an objective vector. Similarly, for the acceleration,

$$\mathbf{a}^* \equiv \frac{\partial^2 \boldsymbol{\chi}^*}{\partial t^2} = \mathbf{Q}\mathbf{a} + 2\dot{\mathbf{Q}}\mathbf{v} + \ddot{\mathbf{Q}}\mathbf{x} + \ddot{\mathbf{c}}.$$

5.4 Material symmetry

Let $\boldsymbol{\sigma}$ be the stress in configuration B_t, and let $\mathbf{F}, \mathbf{F}'$ be the deformation gradients in B_t relative to the reference configurations $\mathrm{B}_r, \mathrm{B}'_r$ respectively, as depicted in Fig. 1. We denote by $\mathbf{g}$ and $\mathbf{g}'$ the response functions relative to B_r and B'_r, respectively, so that

$$\boldsymbol{\sigma} = \mathbf{g}(\mathbf{F}) = \mathbf{g}'(\mathbf{F}'). \tag{5.11}$$

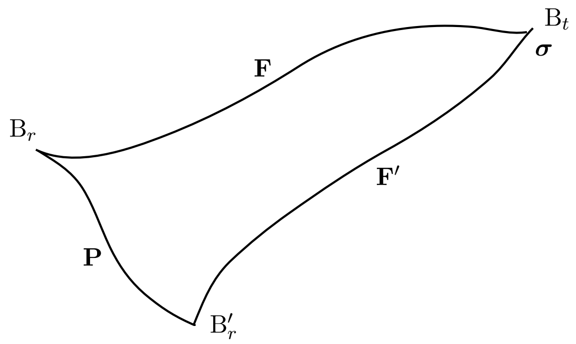


Figure 1: Paths of deformation with deformation gradients $\mathbf{F}$ and $\mathbf{F}'$ from reference configurations B_r and B'_r, which are connected by deformation gradient $\mathbf{P}$.

Let $\mathbf{P} = \operatorname{Grad} \mathbf{X}'$ be the deformation gradient of B'_r relative to B_r, where $\mathbf{X}'$ is the position vector of a point in B'_r. Then

$$\mathbf{F} = \mathbf{F}'\mathbf{P}. \tag{5.12}$$

To prove (5.12), we use index notation. We have

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \frac{\partial x_i}{\partial X'_k}\frac{\partial X'_k}{\partial X_j} = F'_{ik}P_{kj}.$$

Substitution of (5.12) into (5.11) then gives

$$\mathbf{g}(\mathbf{F}'\mathbf{P}) = \mathbf{g}'(\mathbf{F}').$$

In general, the response of the material relative to B'_r differs from that relative to B_r, i.e. $\mathbf{g}' \neq \mathbf{g}$. However, for certain $\mathbf{P}$ we may have $\mathbf{g}' = \mathbf{g}$, in which case

$$\mathbf{g}(\mathbf{F}'\mathbf{P}) = \mathbf{g}(\mathbf{F}') \tag{5.13}$$

for all deformation gradients $\mathbf{F}'$ and for all such $\mathbf{P}$.

The set of tensors $\mathbf{P}$ for which (5.13) holds defines the *symmetry of the material relative to* B_r — the larger the set the more symmetry there is. An example is provided by the structure of a cubic crystal, which has certain rotational symmetry.

Let $\mathcal{G}$ denote the set of (invertible) second-order tensors, denoted $\mathbf{H}$, such that, in line with (5.13),

$$\mathbf{g}(\mathbf{FH}) = \mathbf{g}(\mathbf{F}) \tag{5.14}$$

for all deformation gradients $\mathbf{F}$.

Then $\mathcal{G}$ is a *multiplicative* group, called the *symmetry group of the material relative to* B_r.

To show this we note that if $\mathbf{H}_1, \mathbf{H}_2 \in \mathcal{G}$ then, by application of (5.14) for different $\mathbf{F}$ and $\mathbf{H}$,

$$\mathbf{g}(\mathbf{FH}_1\mathbf{H}_2) = \mathbf{g}(\mathbf{FH}_1) = \mathbf{g}(\mathbf{F})$$

and hence $\mathbf{H}_1\mathbf{H}_2 \in \mathcal{G}$ (closure); if $\mathbf{H} \in \mathcal{G}$ then

$$\mathbf{g}(\mathbf{FH}^{-1}) = \mathbf{g}(\mathbf{FH}^{-1}\mathbf{H}) = \mathbf{g}(\mathbf{F})$$

and hence $\mathbf{H}^{-1} \in \mathcal{G}$ (inverse); and $\mathbf{I} \in \mathcal{G}$ since $\mathbf{g}(\mathbf{FI}) = \mathbf{g}(\mathbf{F})$ (identity). Thus, the requirements of closure, inverse and identity for a group are satisfied.

5.4.1 Important example: isotropy

If $\mathcal{G}$ is the proper orthogonal group then the material is said to be *isotropic relative to* B_r, and

$$\mathbf{g}(\mathbf{FQ}) = \mathbf{g}(\mathbf{F}) \tag{5.15}$$

for all proper orthogonal $\mathbf{Q}$ (for every deformation gradient $\mathbf{F}$). Physically, this means that the response of a small specimen of material cut from B_r is independent of its orientation in B_r.

5.4.2 Example: Noll's rule

In general, the symmetry group changes with a change in reference configuration. Let $\mathbf{P}$ be the deformation gradient $\mathrm{B}_r \to \mathrm{B}'_r$. If $\mathcal{G}$ is the symmetry group of the material relative to B_r and $\mathcal{G}'$ that relative to B'_r then

$$\mathcal{G}' = \mathbf{P}\mathcal{G}\mathbf{P}^{-1}. \tag{5.16}$$

To show this let $\mathbf{g}$ and $\mathbf{g}'$ be the response functions relative to B_r and B'_r respectively. Then

$$\begin{aligned} \mathbf{g}'(\mathbf{F}') = \mathbf{g}(\mathbf{F}'\mathbf{P}) &= \mathbf{g}(\mathbf{F}'\mathbf{PH}) \qquad (\text{since } \mathbf{H} \in \mathcal{G}) \\ &= \mathbf{g}(\mathbf{F}'\mathbf{PHP}^{-1}\mathbf{P}) = \mathbf{g}'(\mathbf{F}'\mathbf{PHP}^{-1}), \end{aligned}$$

and hence $\mathbf{PHP}^{-1} \in \mathcal{G}'$, i.e. $\mathbf{H} \in \mathcal{G}$ if and only if $\mathbf{PHP}^{-1} \in \mathcal{G}'$. Equation (5.16) is known as Noll's rule.

If $\mathbf{P}$ is a rotation and $\mathcal{G}$ corresponds to isotropy, then $\mathcal{G}' = \mathcal{G}$. We now focus on isotropic materials, for which purpose we require some further results from tensor algebra.

5.5 Isotropic functions of a second-order tensor

Definition 2. *The scalar function $\phi(\mathbf{T})$ of a symmetric second-order tensor $\mathbf{T}$ is said to be an isotropic function of $\mathbf{T}$ if*

$$\phi(\mathbf{Q}\mathbf{T}\mathbf{Q}^T) = \phi(\mathbf{T}) \tag{5.17}$$

for all orthogonal tensors $\mathbf{Q}$.

We remark that the notion of an isotropic *function* is different from that of an isotropic *tensor*.

An isotropic scalar-valued function of $\mathbf{T}$ is also called a *scalar invariant* of $\mathbf{T}$. We may check that the principal invariants I_1, I_2, I_3 of $\mathbf{T}$ are scalar invariants in accordance with (5.17), as follows. For example, for $I_1 = \operatorname{tr}(\mathbf{T})$ we have already seen that, since $\mathbf{Q}$ is orthogonal,

$$I_1(\mathbf{Q}\mathbf{T}\mathbf{Q}^T) = \operatorname{tr}(\mathbf{Q}\mathbf{T}\mathbf{Q}^T) = \operatorname{tr}(\mathbf{Q}^T\mathbf{Q}\mathbf{T}) = \operatorname{tr}(\mathbf{T}) = I_1(\mathbf{T}),$$

while, for $I_3 = \det(\mathbf{T})$,

$$I_3(\mathbf{Q}\mathbf{T}\mathbf{Q}^T) = \det(\mathbf{Q}\mathbf{T}\mathbf{Q}^T) = (\det\mathbf{Q})(\det\mathbf{T})(\det\mathbf{Q}^T) = \det\mathbf{T} = I_3(\mathbf{T}),$$

and similarly for I_2.

Theorem 1. *$\phi(\mathbf{T})$ is a scalar invariant if and only if it is expressible as a function of I_1, I_2, I_3.*

Proof. It is sufficient to consider $\mathbf{T}$ written in spectral form

$$\mathbf{T} = \sum_{i=1}^{3} t_i \mathbf{e}'_i \otimes \mathbf{e}'_i,$$

where t_i are the eigenvalues of $\mathbf{T}$. Since $\phi(\mathbf{Q}\mathbf{T}\mathbf{Q}^T) = \phi(\mathbf{T})$ for arbitrary orthogonal $\mathbf{Q}$ we conclude that ϕ depends on $\mathbf{T}$ only through its eigenvalues, and we may write

$$\phi(\mathbf{T}) \equiv \phi(t_1, t_2, t_3)$$

with

$$\phi(t_1, t_2, t_3) = \phi(t_{\sigma_1}, t_{\sigma_2}, t_{\sigma_3})$$

for all $(\sigma_1\sigma_2\sigma_3)$ permutations of (123). But t_1, t_2, t_3 are the roots of the cubic

$$t^3 - I_1 t^2 + I_2 t - I_3 = 0$$

and are therefore functions of the principal scalar invariants I_1, I_2, I_3. Hence ϕ depends only on I_1, I_2, I_3 or, equivalently, it depends symmetrically on t_1, t_2, t_3.

Let $\mathbf{G}(\mathbf{T})$ be a symmetric second-order tensor function of $\mathbf{T}$. □

Definition 3. $\mathbf{G}(\mathbf{T})$ *is said to be an isotropic tensor function of* $\mathbf{T}$ *if*

$$\mathbf{G}(\mathbf{Q}\mathbf{T}\mathbf{Q}^T) = \mathbf{Q}\mathbf{G}(\mathbf{T})\mathbf{Q}^T \tag{5.18}$$

for all orthogonal $\mathbf{Q}$.

A specific example of a function satisfying (5.18) is as follows. Let $\phi_0, \phi_1, \ldots, \phi_N$ be scalar invariants of $\mathbf{T}$. Then

$$\mathbf{G}(\mathbf{T}) = \phi_0\mathbf{I} + \phi_1\mathbf{T} + \cdots + \phi_N\mathbf{T}^N$$

is an isotropic function of $\mathbf{T}$.

Theorem 2. *If* $\mathbf{G}(\mathbf{T})$ *is isotropic then its eigenvalues are scalar invariants of* $\mathbf{T}$.

Proof. Let $\alpha(\mathbf{T})$ be a principal value of $\mathbf{G}(\mathbf{T})$, so that

$$\det[\mathbf{G}(\mathbf{T}) - \alpha(\mathbf{T})\mathbf{I}] = 0.$$

Similarly, $\alpha(\mathbf{Q}\mathbf{T}\mathbf{Q}^T)$ is the corresponding principal value of $\mathbf{G}(\mathbf{Q}\mathbf{T}\mathbf{Q}^T)$, so that

$$\det[\mathbf{G}(\mathbf{Q}\mathbf{T}\mathbf{Q}^T) - \alpha(\mathbf{Q}\mathbf{T}\mathbf{Q}^T)\mathbf{I}] = 0.$$

Using (5.18) we may re-write this as

$$\det[\mathbf{Q}\mathbf{G}(\mathbf{T})\mathbf{Q}^T - \alpha(\mathbf{Q}\mathbf{T}\mathbf{Q}^T)\mathbf{I}] = 0,$$

and hence, noting that this may also be written

$$(\det\mathbf{Q})\det[\mathbf{G}(\mathbf{T}) - \alpha(\mathbf{Q}\mathbf{T}\mathbf{Q}^T)\mathbf{I}](\det\mathbf{Q}^T) = 0,$$

we deduce that

$$\det[\mathbf{G}(\mathbf{T}) - \alpha(\mathbf{Q}\mathbf{T}\mathbf{Q}^T)\mathbf{I}] = 0.$$

Thus, $\alpha(\mathbf{Q}\mathbf{T}\mathbf{Q}^T)$ is a principal value of $\mathbf{G}(\mathbf{T})$ *for all orthogonal* $\mathbf{Q}$ and hence

$$\alpha(\mathbf{Q}\mathbf{T}\mathbf{Q}^T) = \alpha(\mathbf{T})$$

for all orthogonal $\mathbf{Q}$, i.e. the principal values are scalar invariants of $\mathbf{T}$.

□

Theorem 3. *Every eigenvector of* $\mathbf{T}$ *is an eigenvector of the isotropic function* $\mathbf{G}(\mathbf{T})$.

Proof. Let t_1, t_2, t_3 be eigenvalues of $\mathbf{T}$ corresponding to (orthonormal) eigenvectors $\mathbf{m}_1$, $\mathbf{m}_2$, $\mathbf{m}_3$. Then

$$\mathbf{T} = t_1\mathbf{m}_1 \otimes \mathbf{m}_1 + t_2\mathbf{m}_2 \otimes \mathbf{m}_2 + t_3\mathbf{m}_3 \otimes \mathbf{m}_3.$$

Suppose that

$$\mathbf{G}(\mathbf{T})\mathbf{m}_1 = \alpha\mathbf{m}_1 + \beta\mathbf{m}_2 + \gamma\mathbf{m}_3.$$

Let $\mathbf{Q}$ be a rotation about $\mathbf{m}_1$ through π, so that

$$\mathbf{Qm}_1 = \mathbf{m}_1, \quad \mathbf{Qm}_2 = -\mathbf{m}_2, \quad \mathbf{Qm}_3 = -\mathbf{m}_3.$$

Then, it follows that

$$\mathbf{QTQ}^T = \mathbf{T}$$

and

$$\begin{aligned} \mathbf{G}(\mathbf{T})\mathbf{m}_1 &= \mathbf{G}(\mathbf{QTQ}^T)\mathbf{m}_1 = \mathbf{QG}(\mathbf{T})\mathbf{Q}^T\mathbf{m}_1 \qquad \text{(by isotropy)} \\ &= \mathbf{QG}(\mathbf{T})\mathbf{m}_1 = \mathbf{Q}(\alpha\mathbf{m}_1 + \beta\mathbf{m}_2 + \gamma\mathbf{m}_3), \end{aligned}$$

i.e.

$$\alpha\mathbf{m}_1 + \beta\mathbf{m}_2 + \gamma\mathbf{m}_3 = \alpha\mathbf{m}_1 - \beta\mathbf{m}_2 - \gamma\mathbf{m}_3,$$

and hence, since $\mathbf{m}_1$, $\mathbf{m}_2$, $\mathbf{m}_3$ are linearly independent, $\beta = \gamma = 0$. Thus, $\mathbf{m}_1$ is an eigenvector of $\mathbf{G}(\mathbf{T})$. Similarly for $\mathbf{m}_2$ and $\mathbf{m}_3$. □

Theorem 4. *A symmetric second-order tensor-valued function* $\mathbf{G}(\mathbf{T})$ *of the second-order symmetric tensor* $\mathbf{T}$ *is isotropic if and only if it has the representation*

$$\mathbf{G}(\mathbf{T}) = \phi_0\mathbf{I} + \phi_1\mathbf{T} + \phi_2\mathbf{T}^2, \tag{5.19}$$

where ϕ_0, ϕ_1, ϕ_2 *are functions of* I_1, I_2, I_3*, i.e. they are scalar invariants of* $\mathbf{T}$*.*

Proof. If (5.19) holds then $\mathbf{G}(\mathbf{T})$ is clearly isotropic. On the other hand, if $\mathbf{G}(\mathbf{T})$ satisfies (5.18) then we need to show that (5.19) follows.

We know from Theorem 3 that $\mathbf{G}(\mathbf{T})$ is coaxial with $\mathbf{T}$, and from Theorem 2 that the principal values of $\mathbf{G}(\mathbf{T})$ are invariants of $\mathbf{T}$. Let t_i and g_i be the principal values of $\mathbf{T}$ and $\mathbf{G}(\mathbf{T})$ and suppose that t_1, t_2, t_3 are distinct. Consider the three equations

$$\phi_0 + \phi_1 t_i + \phi_2 t_i^2 = g_i, \qquad i \in \{1, 2, 3\}, \tag{5.20}$$

for the three unknowns ϕ_0, ϕ_1, ϕ_2. The solutions ϕ_0, ϕ_1, ϕ_2 are functions of t_i, g_i $(i = 1, 2, 3)$ which, from Theorems 1 and 2, are themselves functions only of I_1, I_2, I_3. Thus, ϕ_0, ϕ_1, ϕ_2 are uniquely defined by (5.20) as invariants of $\mathbf{T}$. Since $\mathbf{G}(\mathbf{T})$ and $\mathbf{T}$ are coaxial equation (5.20) is just (5.19) referred to principal axes. Hence, by multiplying equation (5.20) by $\mathbf{m}_i \otimes \mathbf{m}_i$ and summing over i, we obtain (5.19). If t_1, t_2, t_3 are not distinct this proof requires modification, but we omit the details.

□

5.6 Isotropic elasticity

From the definition (5.15) of isotropy we have

$$\boldsymbol{\sigma} = \mathbf{g}(\mathbf{F}) = \mathbf{g}(\mathbf{FQ}) \tag{5.21}$$

for all proper orthogonal $\mathbf{Q}$ and each deformation gradient $\mathbf{F}$.

The choice $\mathbf{Q} = \mathbf{R}^T$ and use of the polar decomposition $\mathbf{F} = \mathbf{VR}$ in (5.21) gives

$$\boldsymbol{\sigma} = \mathbf{g}(\mathbf{V}). \tag{5.22}$$

Next, objectivity of the constitutive law $\boldsymbol{\sigma} = \mathbf{g}(\mathbf{F})$ yields

$$\mathbf{Qg}(\mathbf{V})\mathbf{Q}^T = \mathbf{Qg}(\mathbf{F})\mathbf{Q}^T = \mathbf{g}(\mathbf{QF})$$

for all proper orthogonal $\mathbf{Q}$, and, with $\mathbf{F}$ replaced by $\mathbf{QF}$ and $\mathbf{Q}$ by $\mathbf{P}$ in (5.21),

$$\mathbf{g}(\mathbf{QF}) = \mathbf{g}(\mathbf{QFP})$$

for all proper orthogonal $\mathbf{P}$ and $\mathbf{Q}$. Hence

$$\mathbf{Qg}(\mathbf{V})\mathbf{Q}^T = \mathbf{g}(\mathbf{QFP}).$$

By choosing $\mathbf{P} = (\mathbf{QR})^T$ and writing $\mathbf{F} = \mathbf{VR}$ we then obtain

$$\mathbf{Qg}(\mathbf{V})\mathbf{Q}^T = \mathbf{g}(\mathbf{QVQ}^T) \tag{5.23}$$

for all orthogonal tensors $\mathbf{Q}$. In fact, since $\mathbf{Q}$ occurs twice on each side of (5.23), allowing $\mathbf{Q}$ to be improper orthogonal does not affect (5.23), which then states that $\mathbf{g}(\mathbf{V})$ is an isotropic function of $\mathbf{V}$ in accordance with the definition (5.18). Thus, the response function $\mathbf{g}$ has all the properties associated with the isotropic tensor function $\mathbf{G}$ discussed in Section 3.5.

In particular, for an *isotropic elastic material*, $\boldsymbol{\sigma} = \mathbf{g}(\mathbf{V})$ *is coaxial with* $\mathbf{V}$, i.e. with the Eulerian principal axes, and, from Theorem 4, we therefore have

$$\boldsymbol{\sigma} = \mathbf{g}(\mathbf{V}) = \phi_0\mathbf{I} + \phi_1\mathbf{V} + \phi_2\mathbf{V}^2, \tag{5.24}$$

where ϕ_0, ϕ_1, ϕ_2 are invariants of $\mathbf{V}$, i.e. functions of

$$i_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad i_2 = \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2, \quad i_3 = \lambda_1\lambda_2\lambda_3.$$

Alternatively, we may write

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)},$$

where

$$\sigma_i = \phi_0 + \phi_1\lambda_i + \phi_2\lambda_i^2 \qquad i = 1, 2, 3.$$

5.7 Hyperelastic materials

Recall that the energy balance equation can be written in the form

$$\int_{R_t} \rho \mathbf{b} \cdot \mathbf{v} dv + \int_{\partial R_t} \mathbf{t} \cdot \mathbf{v} da = \frac{d}{dt} \int_{R_t} \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} dv + \int_{R_t} \boldsymbol{\sigma} \cdot \mathbf{D} dv. \tag{5.25}$$

If there is no dissipation then the work done by the body and surface forces is converted into kinetic energy and stored elastic energy. In this connection an interpretation for the second term on the right-hand side of (5.25) is needed.

Write

$$\int_{R_t} \boldsymbol{\sigma} \cdot \mathbf{D} dv = \int_{R_r} \mathbf{S}^T \cdot \dot{\mathbf{F}} dV,$$

$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}$ being the engineering stress. Then, the integrand $\mathbf{S}^T \cdot \dot{\mathbf{F}}$ is interpreted as the rate of increase of elastic energy per unit volume in B_r.

This prompts the introduction of the *elastic stored energy* $W(\mathbf{F})$ per unit volume in B_r[1] such that

$$\frac{\partial}{\partial t} W(\mathbf{F}) = \mathbf{S}^T \cdot \dot{\mathbf{F}}. \tag{5.26}$$

Hence

$$\int_{R_t} \boldsymbol{\sigma} \cdot \mathbf{D} dv = \int_{R_r} \frac{\partial}{\partial t} W(\mathbf{F}) dV = \frac{d}{dt} \int_{R_r} W(\mathbf{F}) dV,$$

where

$$\int_{R_r} W(\mathbf{F}) dV$$

is the total elastic strain energy in the region R_r. The right-hand side of (5.25) can now be written as

$$\frac{d}{dt}(\text{kinetic energy} + \text{strain energy}).$$

Since W depends only on $\mathbf{F}$, we have

$$\frac{\partial}{\partial t} W(\mathbf{F}) = \frac{\partial W}{\partial F_{ij}} \frac{\partial F_{ij}}{\partial t} = \operatorname{tr}\left(\frac{\partial W}{\partial \mathbf{F}} \dot{\mathbf{F}}\right) = \left(\frac{\partial W}{\partial \mathbf{F}}\right)^T \cdot \dot{\mathbf{F}}, \tag{5.27}$$

where $\dfrac{\partial W}{\partial \mathbf{F}}$ is the second-order tensor with components defined by the convention

$$\left(\frac{\partial W}{\partial \mathbf{F}}\right)_{ji} = \frac{\partial W}{\partial F_{ij}}.$$

Comparison of this with (5.26) and (5.27) shows that

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}, \quad S_{ij} = \frac{\partial W}{\partial F_{ji}}, \tag{5.28}$$

[1] $W(\mathbf{F})$ is also called the *strain energy* or *potential energy* (per unit volume in B_r).

which provides a formula for $\boldsymbol{\sigma}$ in terms of $W(\mathbf{F})$:

$$\boldsymbol{\sigma} = \mathbf{g}(\mathbf{F}) = (\det \mathbf{F})^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}}. \tag{5.29}$$

We remark that $W(\mathbf{F})$ represents the work done (per unit volume at $\mathbf{X}$) by the stress in deforming the material from B_r to B_t (i.e. from $\mathbf{I}$ to $\mathbf{F}$) and is independent of the path taken in deformation space. An elastic material which possesses a strain-energy function W is said to be a *hyperelastic* or *Green elastic* material.

We now write

$$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma} = (\det \mathbf{F})\mathbf{F}^{-1}\mathbf{g}(\mathbf{F}) \equiv \mathbf{h}(\mathbf{F}), \tag{5.30}$$

which defines $\mathbf{h}$, the response function associated with $\mathbf{S}$ (relative to B_r).

Objectivity implies that

$$\mathbf{h}(\mathbf{QF}) = \det(\mathbf{QF})(\mathbf{QF})^{-1}\mathbf{g}(\mathbf{QF}) = (\det \mathbf{F})\mathbf{F}^{-1}\mathbf{g}(\mathbf{F})\mathbf{Q}^T = \mathbf{h}(\mathbf{F})\mathbf{Q}^T$$

for all proper orthogonal $\mathbf{Q}$. On the other hand, material isotropy implies that

$$\mathbf{h}(\mathbf{FQ}^T) = \det(\mathbf{FQ}^T)(\mathbf{FQ}^T)^{-1}\mathbf{g}(\mathbf{FQ}^T) = \mathbf{Q}(\det \mathbf{F})\mathbf{F}^{-1}\mathbf{g}(\mathbf{F}) = \mathbf{Qh}(\mathbf{F})$$

for all orthogonal $\mathbf{Q}$. For an isotropic material the polar decompositions $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$ yields

$$\mathbf{h}(\mathbf{F}) = \mathbf{R}^T\mathbf{h}(\mathbf{V}) = \mathbf{h}(\mathbf{U})\mathbf{R}^T, \tag{5.31}$$

by which, with the aid of (5.24) and the identity $\mathbf{V} = \mathbf{RUR}^T$,

$$\begin{aligned}
\mathbf{h}(\mathbf{U}) &= \mathbf{h}(\mathbf{F})\mathbf{R} = (\det \mathbf{F})\mathbf{F}^{-1}\mathbf{g}(\mathbf{F})\mathbf{R} \\
&= (\det \mathbf{U})\mathbf{U}^{-1}\mathbf{R}^T(\phi_0\mathbf{I} + \phi_1\mathbf{V} + \phi_2\mathbf{V}^2)\mathbf{R} \\
&= (\det \mathbf{U})\mathbf{U}^{-1}(\phi_0\mathbf{I} + \phi_1\mathbf{U} + \phi_2\mathbf{U}^2) = \det \mathbf{U}\,(\phi_0\mathbf{U}^{-1} + \phi_1\mathbf{I} + \phi_2\mathbf{U}),
\end{aligned}$$

where ϕ_i $(i = 0, 1, 2)$ are scalar invariants of $\mathbf{U}$.

Objectivity of W

Since W is a scalar function objectivity requires that it is unaffected by a superimposed rigid-body rotation after deformation, i.e.

$$W(\mathbf{QF}) = W(\mathbf{F}) \tag{5.32}$$

for all rotations $\mathbf{Q}$ for each deformation gradient $\mathbf{F}$. This may also be expressed by referring to W as being indifferent to observer transformations.

Isotropy of W

For a hyperelastic material which is isotropic relative to B_r, $W(\mathbf{F})$ is unaffected by rotations in B_r (prior to deformation). Thus,

$$W(\mathbf{FP}) = W(\mathbf{F}) \tag{5.33}$$

for all rotations $\mathbf{P}$.

Setting $\mathbf{P} = \mathbf{R}^T, \mathbf{F} = \mathbf{VR}$ in (5.33) gives

$$W(\mathbf{F}) = W(\mathbf{V}).$$

Hence, using (5.32) and (5.33),

$$W(\mathbf{QFP}) = W(\mathbf{FP}) = W(\mathbf{F}) = W(\mathbf{V}),$$

and setting $\mathbf{P} = (\mathbf{QR})^T$ then yields

$$W(\mathbf{QVQ}^T) = W(\mathbf{V}) \tag{5.34}$$

for all orthogonal $\mathbf{Q}$. Equation (5.34) states that W is an isotropic scalar function of $\mathbf{V}$ in accordance with the definition (5.17).

Thus, we may regard W as a function of the principal invariants i_1, i_2, i_3 of $\mathbf{V}$ or, equivalently, as a symmetric function of the principal stretches $\lambda_1, \lambda_2, \lambda_3$. In particular, we have

$$W(\lambda_1, \lambda_2, \lambda_3) = W(\lambda_{\sigma_1}, \lambda_{\sigma_2}, \lambda_{\sigma_3}) \tag{5.35}$$

for all $(\sigma_1\sigma_2\sigma_3)$ permutations of (123) and $\lambda_1, \lambda_2, \lambda_3 \in (0, \infty)$.

Mathematically, there is no restriction so far other than (5.35) on the form that the function W may take, but the predictions of material behaviour based on the form of W must make mathematical sense and must also be compatible with what is observed for real materials.

It is usual to take W to be measured from the reference configuration B_r, so that

$$W(1,1,1) = 0. \tag{5.36}$$

Furthermore, if the reference configuration is stress free then we also have the restriction $\mathbf{h}(\mathbf{I}) = \mathbf{O}$ or, in terms of the derivatives of W with respect to the stretches,

$$\frac{\partial W}{\partial \lambda_i}(1,1,1) = 0, \quad i = 1, 2, 3. \tag{5.37}$$

5.8 Stress-deformation relations for an isotropic hyperelastic material

For an isotropic material the strain-energy function is expressible as a function of the principal stretches, as in (5.35). It follows that

$$\dot{W} = \sum_{i=1}^{3} \frac{\partial W}{\partial \lambda_i} \dot{\lambda}_i. \tag{5.38}$$

But, from (5.26),

$$\dot{W} = J\boldsymbol{\sigma} \cdot \mathbf{D} = \mathbf{S}^T \cdot \dot{\mathbf{F}}. \tag{5.39}$$

Also, for an isotropic material, $\boldsymbol{\sigma}$ is coaxial with $\mathbf{V}$ and can be written in the spectral form

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}. \tag{5.40}$$

Equation (5.39) can therefore be expressed as

$$\dot{W} = J \sum_{i=1}^{3} \sigma_i D_{ii}, \tag{5.41}$$

where D_{ii} are the normal components of $\mathbf{D}$ referred to the axes $\mathbf{v}^{(i)}$. In order to obtain expressions for the principal stresses σ_i in terms of the derivatives of W with respect to the stretches we must compare (5.38) with (5.41). First, we need an expression for the components D_{ii}.

Note that by using the identity $\dot{\mathbf{F}} = \mathbf{LF}$ and the polar decomposition $\mathbf{F} = \mathbf{RU}$, the symmetric part of the velocity gradient $\mathbf{D}$ may be written in the form

$$\mathbf{D} = \frac{1}{2}\mathbf{R}(\dot{\mathbf{U}}\mathbf{U}^{-1} + \mathbf{U}^{-1}\dot{\mathbf{U}})\mathbf{R}^T,$$

and that $\mathbf{U}$ has the spectral decomposition

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)},$$

from which it follows that

$$\dot{\mathbf{U}} = \sum_{i=1}^{3} (\dot{\lambda}_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)} + \lambda_i \mathbf{u}^{(i)} \otimes \dot{\mathbf{u}}^{(i)} + \lambda_i \dot{\mathbf{u}}^{(i)} \otimes \mathbf{u}^{(i)}).$$

Using the connection $\mathbf{v}^{(i)} = \mathbf{R}\mathbf{u}^{(i)}$ we calculate the components

$$\begin{aligned} D_{ii} \equiv \mathbf{v}^{(i)} \cdot (\mathbf{D}\mathbf{v}^{(i)}) &= \frac{1}{2}\mathbf{u}^{(i)} \cdot [(\dot{\mathbf{U}}\mathbf{U}^{-1} + \mathbf{U}^{-1}\dot{\mathbf{U}})\mathbf{u}^{(i)}] \\ &= \mathbf{u}^{(i)} \cdot (\dot{\mathbf{U}}\mathbf{U}^{-1}\mathbf{u}^{(i)}) = \mathbf{u}^{(i)} \cdot (\dot{\mathbf{U}}\lambda_i^{-1}\mathbf{u}^{(i)}) \\ &= \lambda_i^{-1}[\mathbf{u}^{(i)} \cdot (\dot{\mathbf{U}}\mathbf{u}^{(i)})] = \lambda_i^{-1}\dot{\lambda}_i, \end{aligned}$$

in which we have used symmetry and the fact that, since $\mathbf{u}^{(i)}$ is a unit vector, $\mathbf{u}^{(i)} \cdot \dot{\mathbf{u}}^{(i)} = 0$. Comparison of (5.38) and (5.41) now gives

$$\sum_{i=1}^{3} \frac{\partial W}{\partial \lambda_i} \dot{\lambda}_i = \sum_{i=1}^{3} J \sigma_i \lambda_i^{-1} \dot{\lambda}_i,$$

and hence

$$J \lambda_i^{-1} \sigma_i = \frac{\partial W}{\partial \lambda_i},$$

i.e.

$$\sigma_i = J^{-1} \lambda_i \frac{\partial W}{\partial \lambda_i}, \quad i = 1, 2, 3, \tag{5.42}$$

where

$$J = \lambda_1 \lambda_2 \lambda_3. \tag{5.43}$$

Expressions for $\mathbf{T}^{(1)}$ and $\mathbf{S}$ analogous to (5.40) can also be obtained. First we note that since $\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}$, $\mathbf{F}^{-1} = \mathbf{U}^{-1}\mathbf{R}^T$, $\mathbf{R}^T\mathbf{v}^{(i)} = \mathbf{u}^{(i)}$ and $\mathbf{U}^{-1}\mathbf{u}^{(i)} = \lambda_i^{-1}\mathbf{u}^{(i)}$ we may write

$$\mathbf{S} = \sum_{i=1}^{3} t_i \mathbf{u}^{(i)} \otimes \mathbf{v}^{(i)}, \tag{5.44}$$

where

$$t_i = J \lambda_i^{-1} \sigma_i = \frac{\partial W}{\partial \lambda_i}. \tag{5.45}$$

Furthermore, from (5.31) we deduce that, for an isotropic material,

$$\mathbf{S} = \mathbf{h}(\mathbf{F}) = \mathbf{h}(\mathbf{U})\mathbf{R}^T \equiv \mathbf{T}^{(1)}\mathbf{R}^T,$$

where we $\mathbf{T}^{(1)}$ is the so-called *Biot stress tensor*. Hence, using (5.44) and (5.45),

$$\mathbf{T}^{(1)} = \sum_{i=1}^{3} t_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}, \tag{5.46}$$

and t_i are just the principal values of $\mathbf{T}^{(1)}$, i.e. the principal Biot stresses. If W is regarded as a function of $\mathbf{U}$ then we may also write

$$\mathbf{T}^{(1)} = \mathbf{S}\mathbf{R} = \left(\frac{\partial W}{\partial \mathbf{F}}\right) \mathbf{R} = \frac{\partial W}{\partial \mathbf{U}}. \tag{5.47}$$

5.9 Stress-deformation relations in terms of the principal invariants of the left stretch tensor V

Instead of using the stretches $\lambda_1, \lambda_2, \lambda_3$ as independent measures of deformation, we now use (equivalently) the principal invariants i_1, i_2, i_3 of the left stretch tensor $\mathbf{V}$. These are defined by

$$i_1 = \mathrm{tr}\,(\mathbf{V}) \equiv \lambda_1 + \lambda_2 + \lambda_3, \tag{5.48}$$

$$i_2 = \frac{1}{2}\left[(\mathrm{tr}\,\mathbf{V})^2 - \mathrm{tr}\,(\mathbf{V}^2)\right] \equiv \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2, \tag{5.49}$$

$$i_3 = \det \mathbf{V} \equiv \lambda_1\lambda_2\lambda_3 \equiv J. \tag{5.50}$$

In this case we write the strain energy as $\tilde{W}(i_1, i_2, i_3)$ and $\mathbf{S}$ may be written

$$\mathbf{S} = \frac{\partial \tilde{W}}{\partial \mathbf{F}} = \tilde{W}_1 \frac{\partial i_1}{\partial \mathbf{F}} + \tilde{W}_2 \frac{\partial i_2}{\partial \mathbf{F}} + \tilde{W}_3 \frac{\partial i_3}{\partial \mathbf{F}}, \tag{5.51}$$

where

$$\tilde{W}_1 = \frac{\partial \tilde{W}}{\partial i_1}, \quad \tilde{W}_2 = \frac{\partial \tilde{W}}{\partial i_2}, \quad \tilde{W}_3 = \frac{\partial \tilde{W}}{\partial i_3}. \tag{5.52}$$

Expressions for $\partial i_1/\partial \mathbf{F}$, $\partial i_2/\partial \mathbf{F}$ and $\partial i_3/\partial \mathbf{F}$ are quite difficult to be computed. We obtain such expressions indirectly by first calculating $\mathbf{T}^{(1)}$ and then using the connection $\mathbf{S} = \mathbf{T}^{(1)}\mathbf{R}^T$ and comparing the result with (5.51).

Using (5.45) we obtain

$$t_i = \tilde{W}_1 + (i_1 - \lambda_i)\tilde{W}_2 + i_3 \lambda_i^{-1} \tilde{W}_3.$$

Hence, the Biot stress is given by

$$\mathbf{T}^{(1)} = \sum_{i=1}^{3} t_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)} = \tilde{W}_1 \mathbf{I} + \tilde{W}_2 (i_1 \mathbf{I} - \mathbf{U}) + i_3 \tilde{W}_3 \mathbf{U}^{-1},$$

and then

$$\mathbf{S} = \mathbf{T}^{(1)}\mathbf{R}^T = \tilde{W}_1 \mathbf{R}^T + \tilde{W}_2 (i_1 \mathbf{R}^T - \mathbf{F}^T) + i_3 \tilde{W}_3 \mathbf{F}^{-1}. \tag{5.53}$$

Comparison of (5.53) with (5.51) shows that

$$\frac{\partial i_1}{\partial \mathbf{F}} = \mathbf{R}^T, \quad \frac{\partial i_2}{\partial \mathbf{F}} = i_1 \mathbf{R}^T - \mathbf{F}^T, \quad \frac{\partial i_3}{\partial \mathbf{F}} = i_3 \mathbf{F}^{-1}. \tag{5.54}$$

In terms of i_1, i_2, i_3 the Cauchy stress tensor has the representation

$$\boldsymbol{\sigma} = i_3^{-1}(\tilde{W}_1 + i_1 \tilde{W}_2)\mathbf{V} - i_3^{-1} \tilde{W}_2 \mathbf{V}^2 + \tilde{W}_3 \mathbf{I}. \tag{5.55}$$

This may be compared directly with (5.24) to provide expressions for the coefficients ϕ_0, ϕ_1, ϕ_2 which appear in (5.24).